4

Strains

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§4.1. Introduction

All of the material covered in the first 3 lectures pertains to *statics*: applied forces induce internal forces, which induce stresses:

$$\boxed{\text{applied forces} \;\Rightarrow\; \text{internal forces} \;\Rightarrow\; \text{stresses}} \tag{4.1}$$

We now go beyond statics into *kinematics*. Stresses produce *deformations* because real materials are not infinitely rigid. Deformations are measured by *strains*. Integration of strains through space gives *displacements*, which measure motions of the particles of the body (structure). As a result the body *changes size and shape*:

$$\boxed{\text{stresses} \;\Rightarrow\; \text{strains} \;\Rightarrow\; \text{displacements} \Rightarrow\; \text{size \& shape changes}} \tag{4.2}$$

Conversely, if the displacements are given as data (as will happen with the Finite Element Method covered in Part IV of this course), one can pass to strains by differentiation, from strains to stresses using material laws such as Hooke's law for elastic materials, and from stresses to internal forces:

$$\boxed{\text{displacements} \Rightarrow \text{strains} \Rightarrow \text{stresses} \Rightarrow \text{internal forces}} \tag{4.3}$$

This lecture focuses on the definition of strains and their connection to displacements. The relation between strains and stresses, which is given by material properties codified into the so-called *constitutive equations*, is studied in the next lecture.

§4.2. Strain: Classification

In general terms, strain is a macroscopic *measure of deformation*. Truesdell and Toupin, in their famous *Classical Field Theories* review article in *Handbuch der Physik*, introduce the concept as

"The change in length and relative direction occasioned by deformation is called, loosely, *strain*." [The term "strain" was introduced by W. J. M. Rankine in 1851.]

The concept is indeed loose until some additional qualifiers are called upon to render the matter more specific.

1. *Average vs. Point*. **Average strain** is that taken over a finite portion of the body; for example using a strain gage or rosette. **Point strain** is obtained by a limit process in which the dimension(s) of the gaged portion is made to approach zero.
2. *Normal vs. Shear*. **Normal strain** measures changes in length along a specific direction. It is also called *extensional strain* as well as *dimensional strain*. **Shear strain** measures changes in angles with respect to two specific directions.
3. *Mechanical vs. Thermal*. **Mechanical strain** is produced by stresses. **Thermal strains** are produced by temperature changes. (The latter are described in the next lecture.)
4. *Finite vs. Infinitesimal*. **Finite strains** are obtained using exact measures of the change in dimensions or angles. **Infinitesimal strains** are obtained by linearizing the finite strain measures with respect to displacement gradients. On account of the nature of this process, infinitesimal

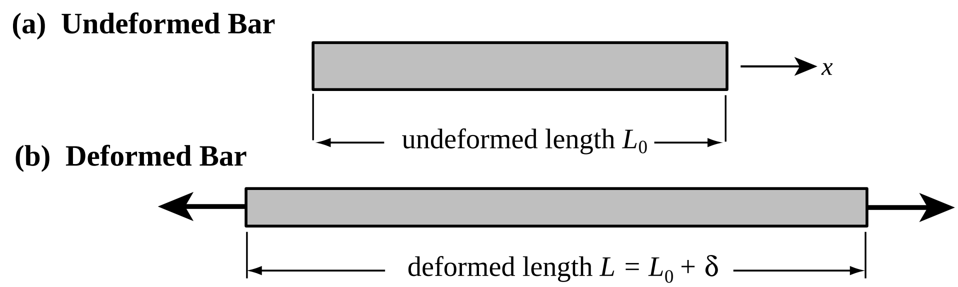

FIGURE 4.1. Undeformed and deformed bar configurations to illustrate average axial (a.k.a. normal, extensional) strain.

strains are also called *linearized strains*. The looser term *small strains* is also found in the literature.

5. *Strain Measures*. For *finite strains* several mathematical measures are in use, often identified with a person name in front. For example, Lagrangian strains, Eulerian strains, Hencky strains, Almansi strains, Murnaghan strains, Biot strains, etc. They have one common feature: as strains get small in the sense that their magnitude is $<< 1$, they coalesce into the infinitesimal (linearized) version. A brief discussion of Lagrangian versus Eulerian strains is provided in §4.3.1 below.

§4.3. Axial (a.k.a. Normal or Extensional) Strains

§4.3.1. Average Strain in 1D

Consider an unloaded bar of length L_0 aligned with the x axis, as shown in Figure 4.1(a). In the literature this is called the *undeformed*, *initial*, *reference*, *original* or *unstretched* configuration. The strains therein are conventionally taken to be zero.

The bar is then pulled by applying an axial force, as shown in Figure 4.1(b). (The undeformed and deformed configurations are shown offset for visualization convenience; in fact both are centered about the x axis.) In this new configuration, called *deformed*, *final*, *current* or *stretched*, its length becomes $L = L_0 + \delta$, where $\delta = L - L_0$ is the bar elongation. The *average axial strain* over the whole bar is defined as

$$\epsilon_{av}^{bar} \stackrel{\text{def}}{=} \frac{L - L_0}{L_{ref}} = \frac{\delta}{L_{ref}} \tag{4.4}$$

Here L_{ref} is the *reference length* selected for the strain computation. The two classical choices are $L_{ref} = L_0$ for *Lagrangian* strains, and $L_{ref} = L$ for *Eulerian* strains. The former is that commonly used in *solid mechanics* and, by extension, structures. The latter is more popular in *fluid mechanics* since a liquid or gas does not retain memory from previous configurations. Whatever the choice, the strain is a *dimensionless* quantity: length divided by length.

If $\delta << L_0$, which leads to the *linearized strain* measure (also called infinitesimal strain or small strain) the choice of L_{ref} makes little difference. For example, suppose $L_0 = 10$ in and $\delta = 0.01$ in, which would be typical of a structural material. Using superscripts L and E for Lagrangian and Eulerian, respectively, we find

$$\epsilon_{av}^{bar,L} = 0.01/10 = 0.10000\%, \qquad \epsilon_{av}^{bar,E} = 0.01/10.01 = 0.09990\%. \tag{4.5}$$

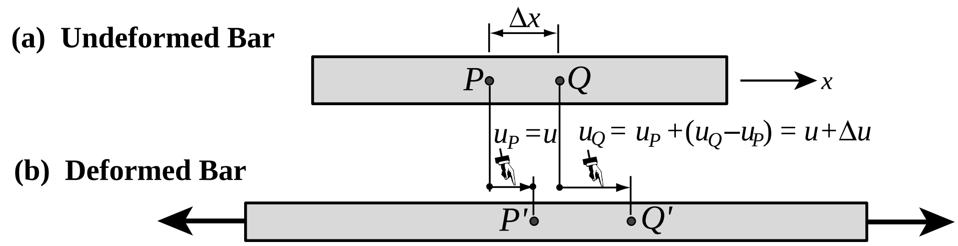

FIGURE 4.2. Undeformed and deformed bar configurations to illustrate point axial strain.

These agree to 3 places. In what follows we will consistently adopt the *Lagrangian* choice, which as noted above is the most common one in solid and structural mechanics.

§4.3.2. Point Strain in 1D

As in the case of stresses covered in Lecture 1, the strain at a point is obtained by a limit process. Consider again the bar of Figure 4.1. In the undeformed configuration mark two coaxial points: P and Q, separated by a small but finite distance Δx, as shown in Figure 4.2(a). (In experimental determination of strains, this is called the *gage length*.) The bar is pulled and moves to the deformed configuration illustrated in Figure 4.2(b). (Undeformed and deformed configurations are again shown offset for visualization convenience.)

Points P and Q move to positions P' and Q', respectively. The *axial displacements* are $u_P = u$ and $u_Q = u_P + (u_Q - u_P) = u + \Delta u$, respectively. The strain at P is obtained by taking the limit of the average strain over Δx as this distance tends to zero:

$$\epsilon_P \stackrel{\text{def}}{=} \lim_{\Delta x \to 0} \frac{(u + \Delta u) - u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}. \tag{4.6}$$

This relation between displacement and point-strains is called a *strain-displacement* equation. It allows one to compute strains directly by differentiation should the displacement variation be known, for example from experimental measurements.

Anticipating the generalization to 3D in §4.3.4: the $\{x, y, z\}$ components of the displacement of P in a 3D body are denoted by u, v and w, respectively. These are actually functions of position, meaning that $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$, in which $\{x, y, z\}$ are the position coordinates of P in the undeformed configuration. Then (4.6) generalizes to

$$\epsilon_{xx} = \frac{\partial u}{\partial x}. \tag{4.7}$$

The generalization of (4.6) to (4.7) involves three changes:

- Point label P is dropped since we can make that a generic (arbitrary) point
- Subscript ‘xx’ is appended to identify the strain component as per rules stated later
- The ordinary derivative du/dx becomes a partial derivative.

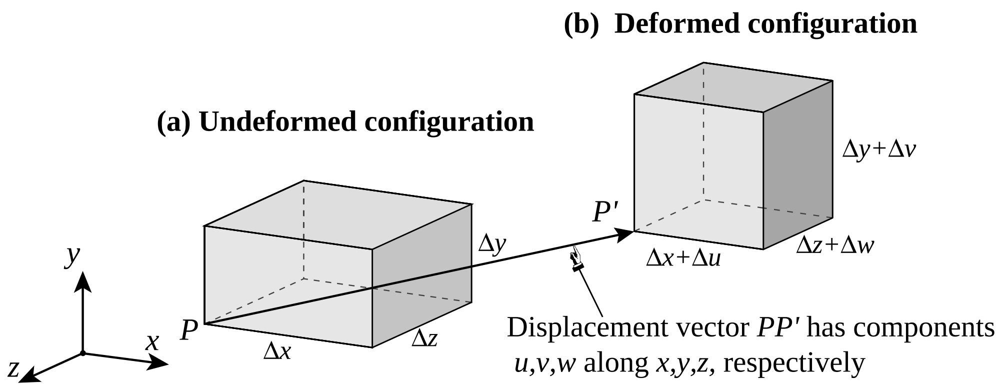

FIGURE 4.3. Undeformed and deformed cube of material in 3D. Shear strains are zero so angles are preserved. Change in cube dimensions grossly exaggerated for visibility.

§4.3.3. Strain Units

As previously noted, strain is dimensionless but in structural mechanics it is often a very small number compared to one. To reduce the number of zeros on the left one can express that number in per-cent, per-mill or "micro" units. For example

$$\epsilon_{xx} = 0.000153 = 0.0153\,\% = 153\,\mu. \tag{4.8}$$

Here μ is the symbol for micros; by definition $1\mu = 10^{-6}$. Sometimes this is written μmm/mm or μin/in, but the unit of length is usually unnecessary.

§4.3.4. Point Normal Strains in 3D

Instead of the bar of Figure 4.1, consider now a small but finite cube of material aligned with the $\{x, y, z\}$ axes, as pictured in Figure 4.3(a). The cube has side dimensions Δx, Δy and Δz, respectively, in the undeformed configuration.

The cube moves to a deformed configuration pictured in Figure 4.3(b). The displacement components are denoted by u, v, and w, respectively. The deformed cube still remains a cube (more precisely, shear strains are assumed to be zero everywhere so angles are preserved) but side lengths change to $\Delta x+\Delta u$, $\Delta y+\Delta v$ and $\Delta z+\Delta w$, respectively. Here Δu, Δv and Δw denote appropriate displacement increments.

The averaged normal strain components are defined as

$$\epsilon_{xx,av} \stackrel{\text{def}}{=} \frac{u+\Delta u-u}{\Delta x} = \frac{\Delta u}{\Delta x},\ \epsilon_{yy,av} \stackrel{\text{def}}{=} \frac{v+\Delta v-v}{\Delta y} = \frac{\Delta v}{\Delta y},\ \epsilon_{zz,av} \stackrel{\text{def}}{=} \frac{w+\Delta w-w}{\Delta z} = \frac{\Delta w}{\Delta z}. \tag{4.9}$$

Point values, at corner P of the cube, are obtained by passing to the limit:

$$\epsilon_{xx} \stackrel{\text{def}}{=} \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x},\quad \epsilon_{yy} \stackrel{\text{def}}{=} \lim_{\Delta y\to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y},\quad \epsilon_{zz} \stackrel{\text{def}}{=} \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z}. \tag{4.10}$$

Of course this process assumes that the indicated limits exist. Components may be tagged with the point at which strains are computed if necessary or advisable. For example: $\epsilon_{xx,P}$, ϵ^{P}_{xx}, or $\epsilon_{xx}(P)$.

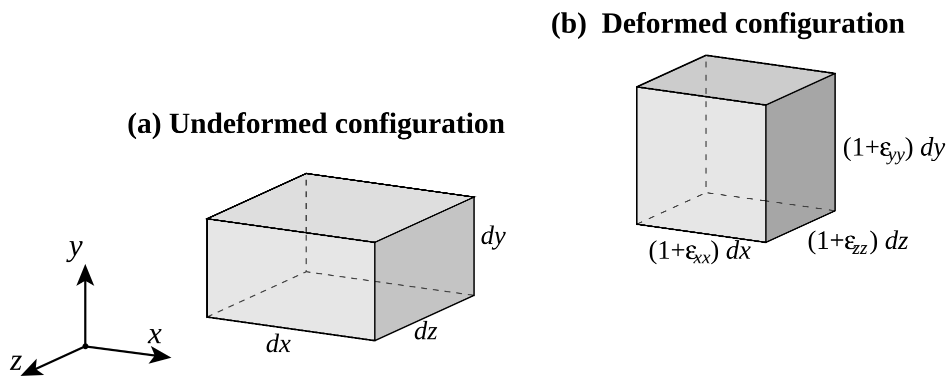

FIGURE 4.4. Slight modification of previous figure to illustrate the concept of volumetric strain.

§4.3.5. Volumetric Strain

Figure 4.4 is a slight modification of the previous one. Here the material cube has now *infinitesimal* dimensions dx, dy and dz. Upon deformation these become $dx + \epsilon_{xx}\, dx = (1+\epsilon_{xx})\, dx$, $dy + \epsilon_{yy}\, dy = (1+\epsilon_{yy})\, dy$, and $dz + \epsilon_{zz}\, dz = (1+\epsilon_{zz})\, dz$. If the cube volume in the undeformed and deformed configurations are denoted by dV_0 and dV, respectively, the change in cube volume is

$$\begin{aligned} dV - dV_0 &= [(1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{yy}) - 1]\, dx\, dy\, dz = \\ &\approx (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})\, dx\, dy\, dz. \end{aligned} \tag{4.11}$$

in which the last simplification assumes that strains are infinitesimal; that is, $\epsilon_{xx} << 1$, etc. The relative volume change is called the *volumetric strain* and is denoted by ϵ_v:

$$\epsilon_v = \frac{dV - dV_0}{dV} = \frac{(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})\, dx\, dy\, dz}{dx\, dy\, dz} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}. \tag{4.12}$$

It can be shown that this quantity is a *strain invariant*, which means that the value does not depend on the choice of axes.

§4.4. Shear Strains

Shear strains measure changes of *angles* as the material distorts in response to shear stress. To define shear strains it is necessary to look at *two* directions that form the plane that undergoes shear distortion. Therefore a one-dimensional view is insufficient to describe what happens. It takes two to shear.

§4.4.1. Average Shear Strains

Figure 4.5(a) shows a cube of material undergoing a pure shear deformation in the $\{x, y\}$ plane. By looking along z we can describe the process through the two-dimensional view of Figure 4.5(b,c). Under the action of the shear stress $\tau_{xy} = \tau_{yx}$, the angle formed by PQ and PR, originally $\pi/2$ radians, becomes $\pi/2 - \gamma$ radians. This change is taken as the definition of the average shear strain associated with directions x and y:

$$\gamma_{xy,av} \stackrel{\text{def}}{=} \gamma. \tag{4.13}$$

By convention $\gamma_{xy,av}$ is positive if the angle $\angle\{PQ, PR\}$ decreases; the motivation being to agree in sign with a positive shear stress.

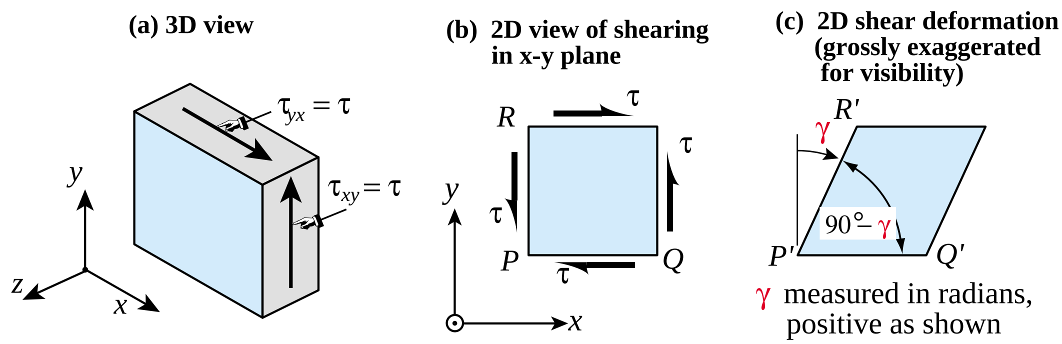

FIGURE 4.5. Average shear strain in $\{x, y\}$ plane.

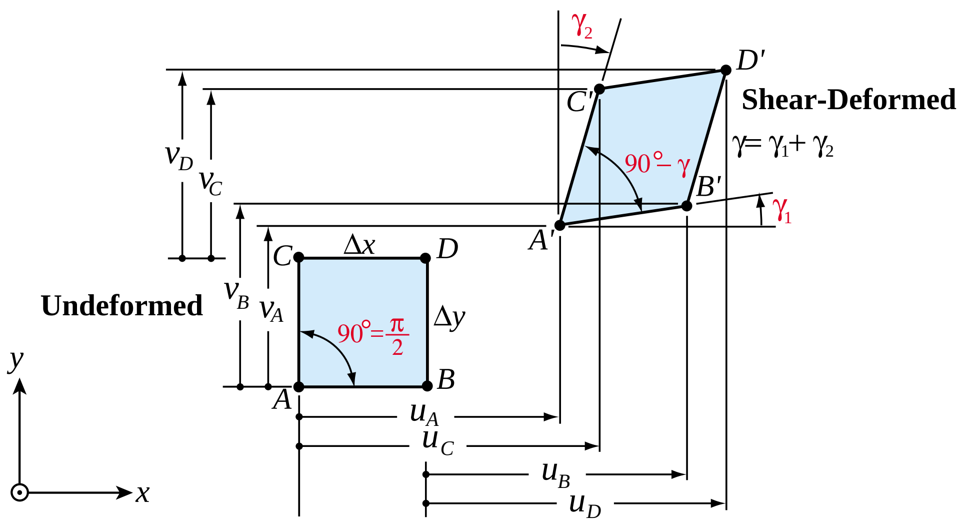

FIGURE 4.6. Computing average shear strain over rectangle ABCD from corner displacement data.

§4.4.2. Connecting Average Shear Strain To Displacements

A rectangle ABCD of side lengths Δx and Δy aligned with x and y respectively, undergoes the shear deformation depicted in Figure 4.6. The original rectangle becomes a parallelogram A'B'C'D', whose sides are not necessarily aligned with the axes. Goal: compute the average shear strain $\gamma_{xy,av} = \gamma$ in terms of the displacements of the four corners.

Observe that $\gamma = \gamma_1 + \gamma_2$, where γ_1 and γ_2 are the angles indicated in Figure 4.6, with positive senses as shown. We have

$$\tan\gamma_1 = \frac{v_{B'} - v_{A'}}{u_{B'} - u_{A'}} = \frac{\Delta v_{BA}}{\Delta x + \Delta u_{BA}}, \qquad \tan\gamma_2 = \frac{u_{C'} - u_{A'}}{v_{C'} - v_{A'}} = \frac{\Delta u_{CA}}{\Delta y + \Delta v_{CA}}. \tag{4.14}$$

We assume that strains are infinitesimal. Consequently $\gamma_1 << 1$ and $\gamma_2 << 1$, whence $\tan\gamma_1 \approx \gamma_1$ and $\tan\gamma_2 \approx \gamma_2$. Likewise, $\Delta u_{BA} << \Delta x$ so $\Delta x + \Delta u_{BA} \approx \Delta x$ in the denominator of $\tan\gamma_1$ and $\Delta v_{CA} << \Delta y$ so $\Delta y + \Delta v_{CA} \approx \Delta y$ in the denominator of $\tan\gamma_2$. Introducing these simplifications into (4.14) yields

$$\gamma_{xy,av} = \gamma = \gamma_1 + \gamma_2 \approx \frac{\Delta v_{BA}}{\Delta x} + \frac{\Delta u_{CA}}{\Delta y} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}. \tag{4.15}$$

§4.4.3. Point Shear Strains in 3D

To define the shear strain γ_{xy} at point P we pass to the limit in the average strain expression (4.15) by shrinking both dimensions Δx and Δy to zero:

$$\gamma_{xy} \stackrel{\text{def}}{=} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \gamma_{xy,av} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} \right). \tag{4.16}$$

In the limit this gives the cross partial-derivative sum

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \tag{4.17}$$

This expression plainly does not change if x and y are reversed, whence $\gamma_{xy} = \gamma_{yx}$ as shown above.

The foregoing limit process can be repeated by taking the angles formed by planes $\{y, z\}$ and $\{z, x\}$ to define $\gamma_{yz,av}$ and $\gamma_{zx,av}$ respectively, followed by passing to the limit as in (4.17). Anticipating the more complete analysis of §4.5, the results are

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{zx}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy}. \tag{4.18}$$

This property: $\gamma_{xy} = \gamma_{yx}$, $\gamma_{yz} = \gamma_{zy}$, and $\gamma_{zx} = \gamma_{xz}$, is called *shear strain reciprocity*. This is entirely analogous to the shear stress reciprocity described in Lecture 1. It follows that the 3D state of strain at a point can be defined by 9 components: 3 extensional and 6 shear, which can be arranged as a 3×3 matrix

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix} \tag{4.19}$$

On account of the shear strain reciprocity property (4.18) the above matrix is *symmetric*. Therefore it can be defined by 6 *independent* components: three normal strains and three shear strains.

§4.5. Strain-Displacement Equations

This section summarizes the connections between displacements and point strains, which have appeared in piecewise manner so far. Consider an arbitrary body in 3D in its undeformed and deformed configurations. A generic point $P(x, y, z)$ moves to $P'(x + u, y + v, z + w)$, in which the displacement components are functions of position:

$$u = u(x, y, z), \qquad v = v(x, y, z), \qquad w = w(x, y, z). \tag{4.20}$$

The components (4.20) define a *displacement field*. For visualization convenience we restrict the foregoing picture to 2D as done in Figure 4.7. At P draw an infinitesimal rectangle PQRS of side lengths $\{dx, dy\}$ aligned with the RCC axes $\{x, y\}$. The square maps to a quadrilateral P'Q'R'S' in the deformed body as illustrated in Figure 4.7(b). To first order in $\{dx, dy\}$ the mapped corner

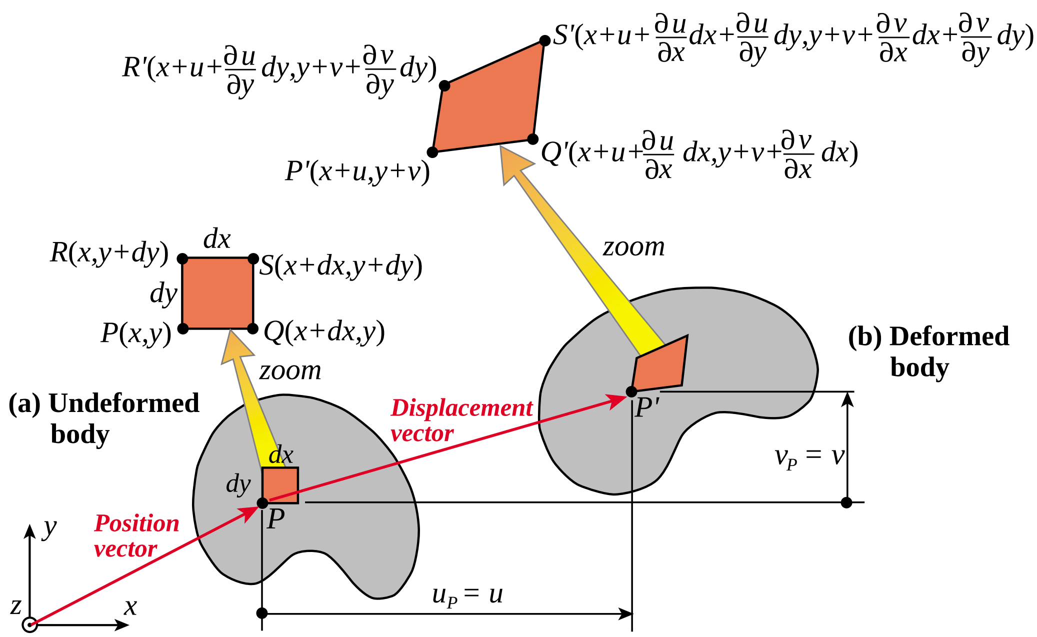

FIGURE 4.7. Connecting strains to displacements: 2D view used for visualization convenience.

points are given by

$$
\begin{array}{rll}
\text{P maps tp P' at} & x+u, & v \\
\text{Q maps to Q' at} & x+u+\dfrac{\partial u}{\partial x}\,dx, & y+v+\dfrac{\partial v}{\partial x}\,dx, \\
\text{R maps to R' at} & x+u+\dfrac{\partial u}{\partial y}\,dy, & y+v+\dfrac{\partial v}{\partial y}\,dy, \\
\text{S maps to S' at} & x+u+\dfrac{\partial u}{\partial x}\,dx+\dfrac{\partial u}{\partial y}\,dy, & y+v+\dfrac{\partial v}{\partial x}\,dx+\dfrac{\partial v}{\partial y}\,dy,
\end{array}
\tag{4.21}
$$

as pictured in Figure 4.7(b).

To express ϵ_{xx} in terms of displacements, take the ratio

$$\epsilon_{xx} = \frac{u_{Q'} - u_{P'}}{dx} = \frac{x+u+\frac{\partial u}{\partial x}\,dx - (x+u)}{dx} = \frac{\partial u}{\partial x}. \tag{4.22}$$

This was derived earlier in §4.3.2 in a 1D context. Notice that *no passing to the limit is necessary here* because we started with *infinitesimal* material elements. Likewise

$$\epsilon_{yy} = \frac{u_{R'} - u_{P'}}{dy} = \frac{y+v+\frac{\partial v}{\partial y}\,dy - (y+v)}{dy} = \frac{\partial v}{\partial y}. \tag{4.23}$$

For the 3D case we get one more normal strain,

$$\epsilon_{zz} = \frac{\partial w}{\partial z}. \tag{4.24}$$

The connection of shear strains to displacement derivatives is more involved. The derivation for γ_{xy} was done in §4.4.2 and §4.4.3, which yields (4.17). The complete result for 3D is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}. \tag{4.25}$$

If the subscripts are switched we get

$$\gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{4.26}$$

Comparing these to (4.25) shows that $\gamma_{xy} = \gamma_{yx}$, etc, which proves the reciprocity property stated in §4.4.3.

§4.6. Displacement Vector Composition

In calculations of joint deflections of truss structures, it is often required to *compose displacement vectors*. The composition rule is *not* the same as that of forces unless the vectors are mutually orthogonal. The difference will be illustrated through a worked out example.

§4.6.1. Deflection of a Truss Structure

The 2-member plane truss shown in Figure 4.8(a) was the subject of Homework Exercise 1.3, using specific values for dimensions, loads and member properties. Here it will be initially processed in symbolic form, keeping members lengths L_{AB}, L_{BC}, load q, angle θ, elastic modulus E (same for both members) and bar area A_b (same for both members) as variables. Objective: find the magnitude of the displacement δ_B of joint B as function of the data. Infinitesimal strains are assumed (this is to be verified after deformations and deflections are computed) and the material obeys Hooke' law.

The first step is to find the internal member forces F_{AB} and F_{BC} from statics. The determination of F_{AB} is done with the FBD shown in Figure 4.8(b), by taking moments with respect to hinge C. (This choice bypasses the need to find reactions R_{Cx} and R_{Cy}, since their contribution to the moment equilibrium equation vanishes.) For moment calculation purposes, the uniform distributed load q may be replaced by a resultant point force $q\,L_{BC}$ at the midpoint of member BC. We get $F_{AB} = q\,L_{BC}/(2\sin\theta)$, as pictured in Figure 4.8(b); this is tension if $F_{AB} > 0$.

Next we find F_{BC} using the FBD drawn in Figure 4.8(c), in which the previously found F_{AB} is used. The applied distributed load is "lumped" to the end joints B and C. Since q is uniform each joint receives one half of the total force, that is, $q\,L_{BC}/2$. Force equilibrium along either x or y yields $F_{BC} = q\,L_{BC}/(2\tan\theta)$, as shown in the figure. [Actually the quickest way to get F_{BC} is to consider equilibrium along the direction CB. Since the projection of $q\,L_{BC}/2$ vanishes, F_{BC} must balance $F_{AB}\cos\theta$ whence $F_{BC} = F_{AB}\cos\theta = q\,L_{BC}\cos\theta/(2\sin\theta) = q\,L_{BC}/(2\tan\theta)$.] Note that both internal forces can be obtained without computing reactions.

Since Hooke's law is assumed to apply and both members are prismatic, their elongations are $\delta_{AB} = F_{AB}L_{AB}/(E\,A_b)$ and $\delta_{BC} = F_{BC}L_{BC}/(E\,A_b)$, respectively, with appropriate signs. See Figure 4.8(d). The graphical composition of these two vectors to get δ_B is detailed in Figure 4.8(e). Mark δ_{AB} and δ_{BC} with origin at B aligned with member directions, pointing away if they are

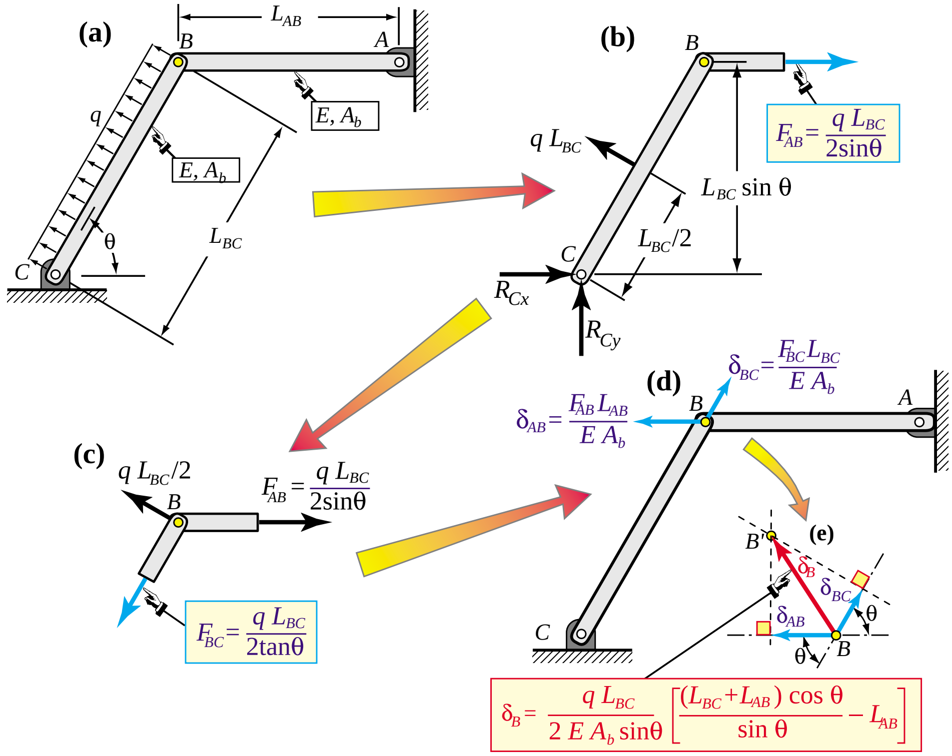

FIGURE 4.8. Computing the motion of a truss node.

positive. Then draw two perpendicular lines from the vector tips. The intersection of those lines gives the deformed position B' of B, and the distance BB' is δ_B. Using trigonometric relations this value can be expressed in terms of the data as

$$\delta_B = \frac{q\,L_{BC}}{2E\,A_b\,\sin\theta}\left(\frac{(L_{BC}+L_{AB})\,\cos\theta}{\sin\theta} - L_{AB}\right). \tag{4.27}$$

The numbers used in Exercise 1.3 were: $L_{AB} = 60$ in, $L_{BC} = 66$ in, $\theta = 60^\circ$ and $q = 80$ lbs/in. In addition to these we take $A_b = 2.5$ in^2 and $E = 30 \times 10^6$ psi (steel). Replacing gives

$$\begin{aligned} &F_{AB} = 3048 \text{ lbs}, \quad F_{BC} = \tfrac{1}{2}\,F_{AB} = 1524 \text{ lbs}, \quad \text{both in tension},\\ &\delta_{AB} = 0.002439 \text{ in}, \quad \delta_{BC} = 0.001341 \text{ in}, \quad \delta_B = 0.003833 \text{ in}. \end{aligned} \tag{4.28}$$

Since these displacements are very small compared to member lengths, the assumption of infinitesimal strains is verified *a posteriori*.

§4.6.2. Forces And Displacements Obey Different Composition Rules

The main point of the foregoing example is to emphasize that displacements do not compose by the same rules as forces. The rules are graphically summarized in Figure 4.9 for two point forces and two displacements in the plane of the figure.

Forces are combined by the well known vector-addition parallelogram rule: the tip of the resultant is at the opposite corner of the parallelogram formed by the two vectors as sides. Displacements

(a) force composition

(b) displacement composition

FIGURE 4.9. Composition of (a) two forces and (b) two displacements acting on the plane of the figure.

are combined by a "cyclic quadrilateral rule" as illustrated. (A cyclic quadrilateral is one that has two opposite right angles.) If the vectors are orthogonal, the composition rules coalesce since in that case both the parallelogram and the cyclic quadrilateral reduce to a rectangle.

If we had 3 or more 2D vectors the rules diverge. Forces can be combined by "chaining" by placing them tail-to-tip. But in general 3 or more displacement vectors with common origin will be incompatible since the perpendicular lines traced from their tips will not usually cross.

The distinction is also important in three-dimensional space. Any number of 3D force vectors can be added by tail-to-tip chaining. Three displacement vectors with common origin can be combined by constructing the normal plane at their tips and finding the point at which the planes intersect. Composing more than three 3D displacement vectors is generally impossible.

An important restriction should be noted. The composition rules illustrated in Figures 4.8(e) and 4.9(b) *apply only to the case of infinitesimal deformations*. This allows displacements to be linearized by Taylor series expansion about the undeformed geometry. For finite displacements see Section 2.4 of Vable's textbook.

Remark 4.1. In advanced courses that cover tensors in arbitrary coordinate systems, it is shown that the displacement field (or, in general, a gradient-generated vector field) transforms as a *covariant* field of order one. On the other hand, a force field (or, in general, a differential-generated vector field) transforms as a *contravariant* field of order one. For the pertinent math see the Wikipedia article

`http://en.wikipedia.org/wiki/Covariance_and_contravariance_of_tensors`