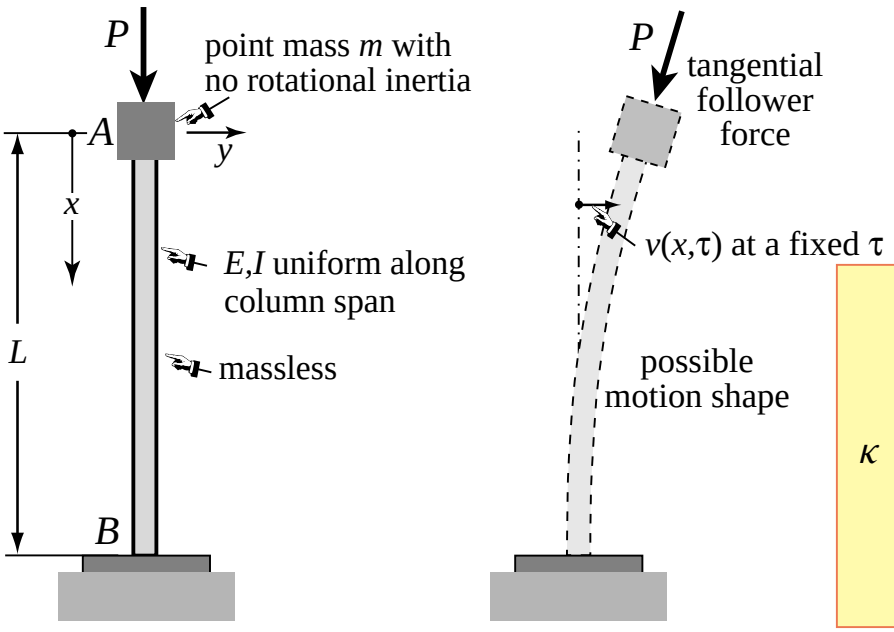
38

Dynamic Stability: Examples

Example in Ch 35: Massless Column With Point Mass @ Top, Subject to Tangential Follower Load



force parametrization:

$$P = \lambda P_{Ecr} = \lambda \frac{\pi^2 EI}{4L^2}$$

Also $k = \sqrt{\dfrac{P}{EI}}$

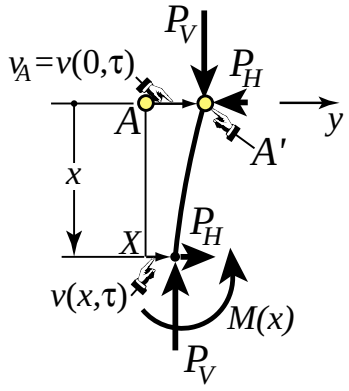
$$\kappa = k L = \sqrt{\frac{P L^2}{EI}} = \frac{\pi}{2}\sqrt{\lambda}$$

$$\lambda = \frac{4\kappa^2}{\pi^2}$$

Massless Column Example: Analytical Solution

Free Body Diagram

Derivation at force residual (equilibrium) level necessary, since there is no TPE



Equation of Motion (EOM):

$$(E\,I\,v'')'' + (P\,v')' = 0 \qquad \text{for} \quad 0 \le x \le L$$

Massless Column Example: Analytical Solution (2)

Solution of EOM is space time separable

$$v(x,\tau) = w(x)\, g(\tau) = \Big(A\, \sin\, kx + B\, \cos\, kx + C\, x + D\Big)\, g(\tau)$$

$$k = \sqrt{\frac{P}{E\,I}}.$$

Boundary conditions (BC)

$$M(0) = 0, \quad V(0) + m\, \ddot{v}(0,\tau) = 0, \quad w(L) = w'(L) = 0$$

BC at top in terms of motion

$$w''(0) = 0$$

$$E\,I\, v'''(0) + m\, \ddot{v}(0,\tau) = E\,I\, w'''(0)\, g(\tau) + m\, w(0)\, \ddot{g}(\tau) = 0$$

Massless Column Example: Analytical Solution (2)

Replace BC into general solution $v(x,\tau) = w(x)\, g(\tau)$ **to get**

$$\begin{bmatrix} 0 & -\kappa^2/L^2 & 0 & 0 \\ -E\,I\,\kappa^3\,g/L^3 & m\,\ddot{g} & 0 & m\,\ddot{g} \\ \sin\kappa & \cos\kappa & L & 1 \\ \kappa\,\cos\kappa/L & -\kappa\,\sin\kappa/L & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

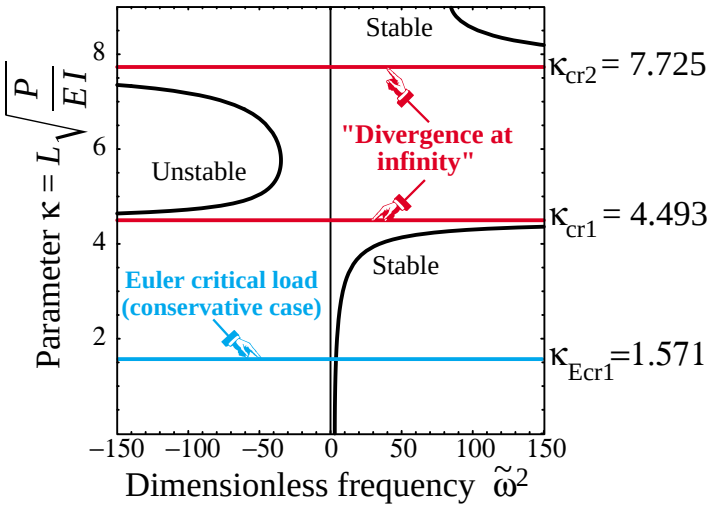
For a nontrivial solution, the determinant of this matrix must be zero, which gives the EOM in the temporal component

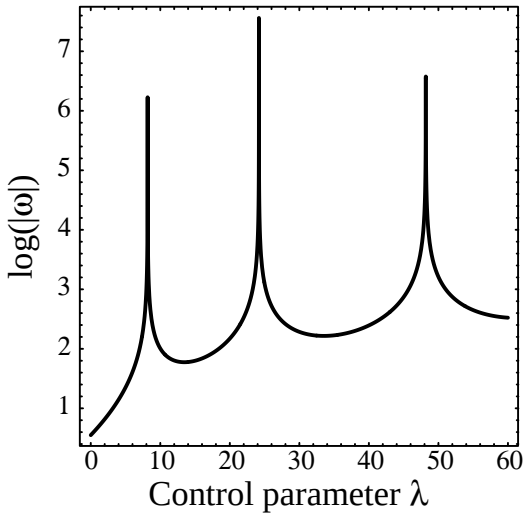
$$\ddot{g} + \omega^2\, g = 0 \qquad \omega^2 = \frac{E\,I}{m\,L^3}\,\frac{\kappa^3}{\sin\kappa - \kappa\cos\kappa}$$

in which

$$k = \frac{P}{E\,I} \qquad \kappa = k\,L = \sqrt{\frac{P\,L^2}{E\,I}} = \frac{\pi}{2}\sqrt{\lambda}$$

Massless Column Example: Analytical Solution (4)



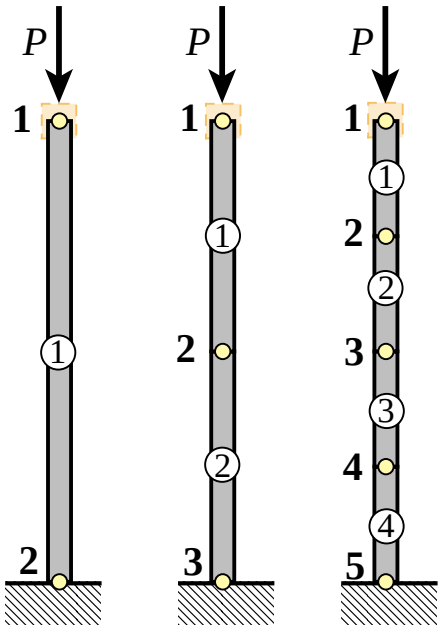


$$P_{cr1} = \frac{\kappa_{cr1}^2 \, E\,I}{L^2} = \frac{20.19072855642663 \, E\,I}{L^2} = 8.18299406375318392 \, P_{Ecr}$$

$$P_{cr2} = \frac{\kappa_{cr2}^2 \, E\,I}{L^2} = \frac{59.679515944109419 \, E\,I}{L^2} = 24.187196778635741 \, P_{Ecr}$$

Massless Column Example: FEM Solution

FEM discretizations:



Massless Column Example: FEM Solution (2)

FEM matrices for one-element discretization:

$$\mathbf{K}_M^e = \frac{E\,I}{L^e}\begin{bmatrix} 12/(L^e)^2 & 6/L^e & -12/(L^e)^2 & 6/L^e \\ 6/L^e & 4 & -6/L^e & 2 \\ -12/(L^e)^2 & -6/L^e & 12/(L^e)^2 & -6/L^e \\ 6/L^e & 2 & -6/L^e & 4 \end{bmatrix}$$

$$\mathbf{K}_G^e = -\frac{P}{30\,L^e}\begin{bmatrix} 36 & 3\,L^e & -36 & 3\,L^e \\ 3\,L^e & 4\,(L^e)^2 & -3\,L^e & -(L^e)^2 \\ -36 & -3\,L^e & 36 & -3\,L^e \\ 3\,L^e & -(L^e)^2 & -3\,L^e & 4\,(L^e)^2 \end{bmatrix}$$

Geometric stiffness from CR formulation

$$\mathbf{K}_L^e = \begin{bmatrix} 0 & -P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{M}^e = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{u}^e = [\,v_1 \quad \theta_1 \quad v_2 \quad \theta_2\,]^T$$

Massless Column Example: FEM Solution (3)

After applying fixed condition at node 2, set up the characteristic eigensystem:

$$(p_i^2\mathbf{M}+\mathbf{K})\mathbf{z}_i = \left(p_i^2 \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{12EI}{L^3} - \frac{6P}{5L} & \frac{6EI}{L^2} - \frac{11P}{10} \\ \frac{6EI}{L^2} - \frac{11P}{10} & \frac{4EI}{L} - \frac{2PL}{15} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ \theta_1 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant of $\mathbf{A} = p_i^2\,\mathbf{M} + \mathbf{K}$ is

$$\det(\mathbf{A}) = \frac{720\,(EI)^2 + L^4\,P\,(3\,P - 8\,m\,L\,p_i^2) + 48\,EI\,L^2\,(P + 5\,m\,L\,p_i^2)}{60\,L^4}$$

Solving for the only eigenvalue:

$$p_1^2 = \frac{720(EI)^2 + 48\,EI\,L^2\,P + 3L^3\,P^2}{8\,m\,L^3\left[PL^2 - 30\,EI\right]}$$

Massless Column Example: FEM Critical Loads Compared to Analytical Solution

Table 33.1 - FEM Results For Massless Column: Dynamic Versus Static

Elements	P_{cr1}^{dyn}	P_{cr2}^{dyn}	P_{cr3}^{dyn}	P_{cr1}^{sta}	P_{cr2}^{sta}	P_{cr3}^{sta}
1	30.0000			2.4859		
2	20.7088	75.1015	197.523	2.4688	22.9462	77.0631
4	20.2322	60.6353	124.886	2.4675	22.2621	62.7526
8	20.1935	59.7481	119.419	2.4674	22.2068	61.7609
16	20.1909	59.6840	118.935	2.4674	22.2066	61.6899
32	20.1907	59.6798	118.902	2.4674	22.2066	61.6853
analytical	20.1907	59.6795	118.852	2.4674	22.2066	61.6850

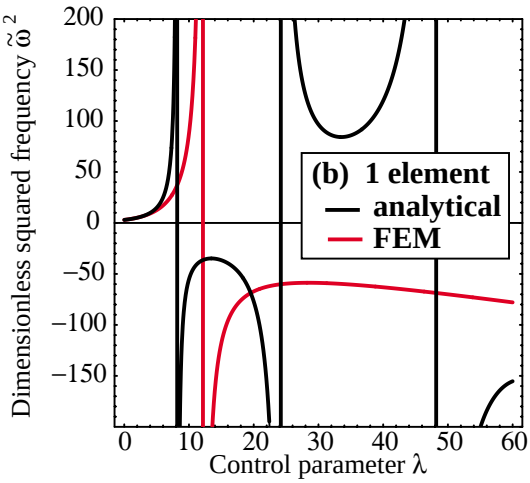
Numbers in table are the coefficients of EI/L^2 for the listed critical loads.

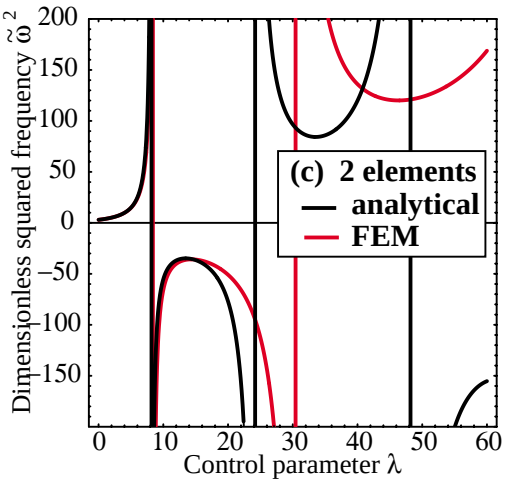
Dynamic critical loads follower force

Static critical loads vertical force

Massless Column Example: FEM Compared to Analytical Solution for Varying Control Parameter

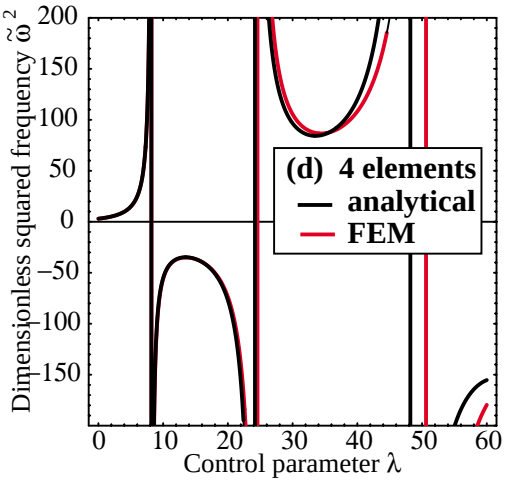
(1 & 2 elements)

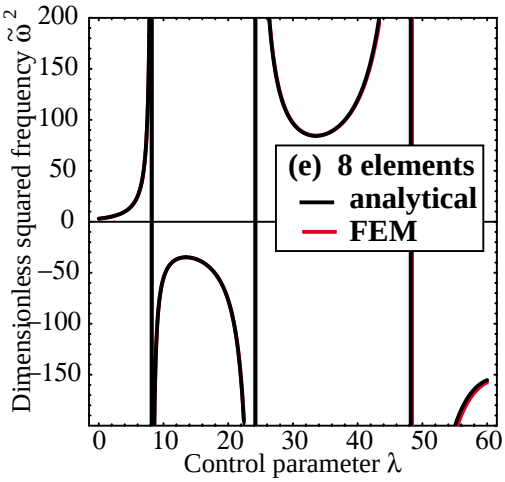




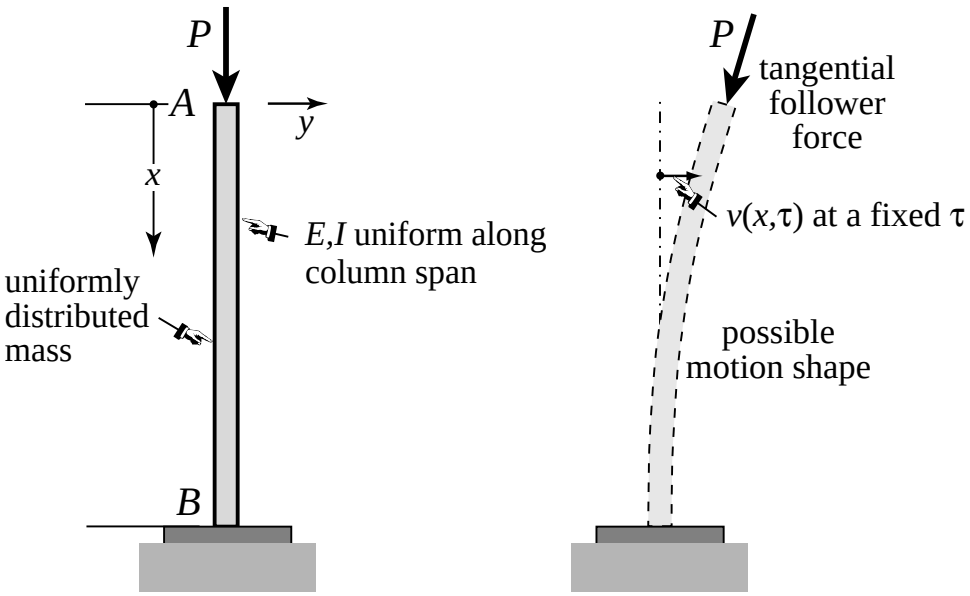
Massless Column Example: FEM Compared to Analytical Solution for Varying Control Parameter

(4 & 8 elements)

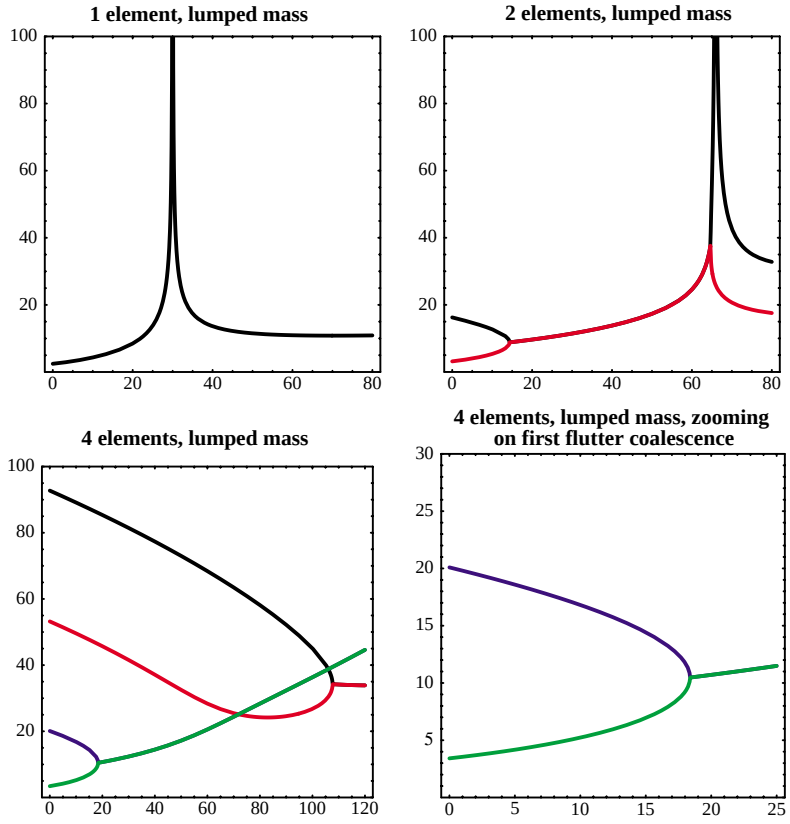




Beck's Column Benchmark



Beck's Column: FEM Solution



Beck's Column: FEM Solution (cont'd)