37

Dynamic Stability: Formulation

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§37.1. Introduction

If the loading is nonconservative the loss of stability may not show up by the system going into another equilibrium state but by going into *unbounded motion*. To encompass this possibility we must consider the *dynamic* behavior of the system because stability is essentially a dynamic concept (recall the definitions in Chapter 27). The following paragraph taken from the excellent book by Bolotin [101] summarizes the subject:

> "The basic method of investigating non-conservative problems in the theory of elastic stability is the *dynamic method*, which is based on the study of the oscillation of the system close to its position of equilibrium. This draws the theory of elastic stability closer to the general theory of stability of motion and its applications in the theory of automatic control, in the hydrodynamics of a viscous liquid and in other fields of applied mechanics and engineering. The *Euler method*, which reduces the problem to an analysis of the bifurcation of the forms of equilibrium of the system, can be looked upon as a particular case of the dynamic method."

The essential steps are as follows. We investigate the motion that occurs after some initial perturbation is applied to the equilibrium state being tested, and from the properties of the motion we can infer or deny stability. It if turns out that the perturbed motion consists of oscillations of increasing amplitude, or is a rapidly increasing departure from the equilibrium state, the equilibrium is unstable; otherwise it is stable.

The practicality of this approach depends crucially on the *linearization* of the equations of motion of the perturbation. Thus we avoid having to trace the ensemble of time histories for every conceivable dynamic departure from equilibrium — which for a system with many degrees of freedom would clearly be a computationally forbidding task.

By linearizing we can express the perturbation motion as the superposition of *complex exponential* elementary solutions. The characteristic exponents of these solutions can be determined through a characteristic value problem or eigenproblem. This problem includes the free-vibration natural frequency eigenproblem as particular case when the system is conservative and the tangent stiffness matrix is symmetric. Through the stability criterion discussed in §37.3, the set of characteristic exponents gives complete information on the linearized stability of the system at the given equilibrium configuration.

In practical studies the characteristic exponents are functions of the control parameter λ. Assuming that the system is stable for sufficiently small λ values, say $\lambda = 0$, we are primarily concerned with finding the first occurrence of λ at which the system loses stability. The transition to instability may occur in two different ways, which receive the names *divergence* and *flutter*, respectively.[1]

The distinction between divergence and flutter instability is important in that the singular-stiffness test discussed in Chapters 27ff *remains valid if the stability loss occurs by divergence*,[2] although of course the tangent stiffness is not necessarily symmetric. Therefore it follows that in that case

[1] These names originated in aeronautical engineering applications. More specifically the investigation of sudden airplane "blow ups" during the period 1920–1935, when the appearance of monoplanes allowed increasing airspeeds. See **Notes and Bibliography**. In the mathematical literature a flutter-like phenomenon goes by the name "Hopf bifurcation" as well as "Poincaré-Andronov-Hpf" bifurcation. See eponymous Wikipedia article for references.

[2] Strictly speaking, it remains valid if divergence occurs by roots passing through zero frequency. The case of *divergence at infinity*, in which roots pass through infinite frequency, is similar to flutter in that it requires the dynamic criterion.

we may fall back upon the static criterion, which is simpler to apply because it does not involve information about mass and damping. Such a regression is not possible, however, if the loss of stability occurs by flutter.

For the convenience of the reader, we reproduce below the stability classification of Chapter 27, with a minor refinement for divergence.

$$\text{Loading}\begin{cases}\text{Conservative}\begin{cases}\text{Static criterion (Euler method): singular stiffness}^*\\ \text{Dynamic criterion: zero frequency}^*\end{cases}\\ \text{Nonconservative}\begin{cases}\text{Dynamic criterion}\begin{cases}\text{Divergence at zero frequency}^*\\ \text{Divergence at infinity frequency}\\ \text{Flutter}\end{cases}\end{cases}\end{cases}$$

Cases marked by * are equivalent in the sense of different methods giving the same results.

§37.2. The Linearized Equations of Motion

A control-state response path is traced. A configuration C in static equilibrium under the control parameter value $\lambda = \lambda_C$ is to be investigated for dynamic equilibrium. See Figure 37.1

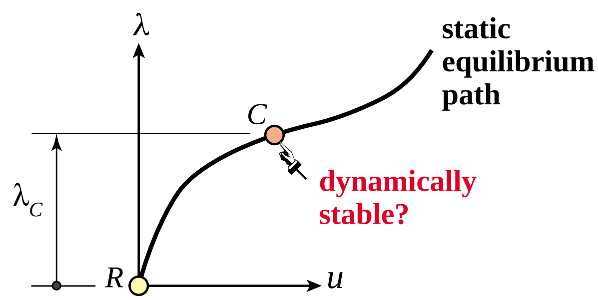

FIGURE 37.1. A configuration C in static equilibrium under value λ of the control parameter is to be investigated for dynamic stability.

The equilibrium state is defined by the state vector $\mathbf{u}$. At time[3] $\tau = 0$ apply a dynamic input (*e.g.*, an impulse) to this configuration and examine the subsequent motion of the system. Roughly speaking if the motion is unbounded (remains bounded) as τ tends to infinity the system is dynamically unstable (stable).

As noted in the Introduction, to simplify the mathematical treatment we consider only the *local stability* condition: the imparted excitation is so tiny that the subsequent motion can be viewed as a linearizable perturbation. We are effectively dealing with *small perturbations* about the equilibrium position, although the latter may be the result of doing static nonlinear analysis.

Let $\mathbf{M}$ be the symmetric mass matrix, which is assumed positive definite, and $\mathbf{K}$ the tangent stiffness matrix, which is real but generally *unsymmetric* because of the nonconservative load stiffness contributions. The perturbed motion is denoted as

$$\mathbf{d}(\tau) = \mathbf{u}(\tau) - \mathbf{u}(0), \quad \text{for } \tau \geq 0^+ \tag{37.1}$$

[3] The symbol τ denotes real time because t is used throughout previous Chapters to denote a pseudotime parameter. Only real time is considered in this Chapter and the next one.

The discrete, unforced, undamped governing equations of motion (EOM) are

$$\mathbf{M}\,\ddot{\mathbf{d}} + \mathbf{K}\,\mathbf{d} = \mathbf{0}, \tag{37.2}$$

in which a superposed dot — unlike previous Chapters — denotes differentiation with respect to *real* time. That is, $\dot{\mathbf{d}} \equiv \partial \mathbf{d}/\partial\tau$. The system of ordinary differential equations (37.2) expresses linearized dynamic balance between stiffness and inertial forces.

Remark 37.1. In structural with rotational DOFs, $\mathbf{M}$ might be only nonnegative definite because of the presence of zero rotational masses. In that case it is assumed that those DOFs have been eliminated by a static condensation process.

The assumption of positive definite mass matrix also excludes the presence of Lagrange multipliers in the state vector $\mathbf{u}$, because the associated masses of such degrees of freedom are zero. Again the stability criteria can be extended by eliminating the multipliers in the linearized equations of motion.

Remark 37.2. We shall ignore damping effects because of two reasons:

(1) The effect of diagonalizable, light viscous structural damping does not generally affect stability results (it certainly does not when stability loss is by divergence).

(2) The effect of more complicated nonlinear damping mechanisms such as dry friction may not be amenable to linearization.

Thus cases when damping effects are significant lead to mathematics beyond the scope of this course. Readers interested in pursuing this topic are referred to the vast literature on the subject of dynamic stability, srating with the references recommended under **Notes and Bibliography**.

§37.3. The Characteristic Problem

The linear ODE system (37.2) can be treated by assuming the eigenmodal expansion

$$\mathbf{d}(\tau) = \sum_i \mathbf{d}_i(\tau) = \sum_i A_i\,\mathbf{z}_i\,e^{p_i\tau}, \tag{37.3}$$

in which index i ranges over the number of degrees of freedom (dimension of state vector). The p_i are generally complex numbers called the *characteristic exponents* whereas the corresponding column vectors $\mathbf{z}_i$ are the *characteristic modes* or *characteristic vectors*.[4]

The coefficients A_i in the solution (37.3) may be determined from the initial conditions (IC) at $\tau = 0$ projected on the eigenvectors $\mathbf{z}_i$. We will see, however, that only the eigenvalues p_i are of interest in stability assessment. In other words, the IC are irrelevant in that regard.

Replacing $\ddot{\mathbf{d}}_i = p_i^2\mathbf{d}$ into (37.2) yields

$$\boxed{(\mathbf{K} + p_i^2\,\mathbf{M})\,\mathbf{z}_i = \mathbf{0},} \tag{37.4}$$

which is the *characteristic problem* or *eigenproblem* that governs linearized dynamic stability.

[4] In his classical treatise [101] Bolotin employs s for what we call here p, and so do other authors. This notation connects well to the common use of the Laplace transform to do more complicated dynamic systems. However, we have already reserved s for Piola-Kirchhoff stresses as well as arclength.

§37.3.1. The Associated Eigenproblem

The characteristic problem (37.4) befits the generalized unsymmetric eigenproblem of linear algebra

$$\mathbf{A}\,\mathbf{x}_i = \mu_i\,\mathbf{B}\,\mathbf{x}_i, \tag{37.5}$$

in which

$$\mathbf{A} = \mathbf{K}, \quad \mathbf{B} = \mathbf{M}, \quad \mu_i = -p_i^2. \tag{37.6}$$

Here matrix $\mathbf{A} = \mathbf{K}$ is real and generally unsymmetric whereas $\mathbf{B} = \mathbf{M}$ is real, symmetric and positive definite. The eigenvalues $\mu_i = -p_i^2$ of this eigenproblem may be real or complex. If the latter, they occur in conjugate pairs. The square roots of the eigenvalues yield the *characteristic exponents* p_i of the eigenmodal expansion (37.3).

§37.3.2. Connection with the Free-Vibration Eigenproblem

If the system is *conservative* and *stable*, $\mathbf{K}$ is symmetric and positive definite. If so all roots p_i^2 of (37.4) are *negative real*. Their square roots are *purely imaginary* numbers:

$$p_i = \pm j\omega_i, \quad \text{in which} \quad j = \sqrt{-1}. \tag{37.7}$$

The nonnegative real numbers ω_i are the *natural frequencies* of free vibration.[5] Since $p^2 = -\omega_i^2$, (37.4) reduces to the usual vibration eigenproblem

$$(\mathbf{K} - \omega_i^2\mathbf{M})\,\mathbf{z}_i = \mathbf{0}, \quad \text{or} \quad \mathbf{K}\,\mathbf{z}_i = \omega_i^2\mathbf{M}\,\mathbf{z}_i. \tag{37.8}$$

It follows that for the stable conservative case we regress to a well studied problem. If so the system will simply *vibrate*, that is, perform *harmonic oscillations* about the equilibrium position because each root is associated with the solution

$$e^{j\omega_i\tau} = \cos\omega_i\tau + j\,\sin\omega_i\tau. \tag{37.9}$$

which is Euler's complex exponential formula. The presence of positive damping will of course damp out these oscillations and the system eventually returns to the static equilibrium position.

§37.4. Characteristic Exponents and Stability

The characteristic exponents will be generally complex numbers:

$$\boxed{p_i = \alpha_i + j\omega_i,} \tag{37.10}$$

in which α_i and ω_i are *real* numbers, and $j = \sqrt{-1}$ denotes the imaginary unit. Accordingly,

$$p_i^2 = (\alpha_i^2 - \omega_i^2) + 2j\alpha_i\,\omega_i, \tag{37.11}$$

Recall that the exponential of a complex number has the real-plus-imaginary form

$$\boxed{e^{p_i\tau} = e^{(\alpha_i + j\omega_i)\tau} = e^{\alpha_i\tau}(\cos\omega_i\tau + j\,\sin\omega_i\tau).} \tag{37.12}$$

On the basis of the representation (37.12) we can classify the growth behavior of the subsequent motion, and consequently the dynamic stability of the system, as examined in the next subsections.

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[5] Also called *circular frequencies*. Their physical units are radians per second. Thay can be converted to frequencies f_i expressed in cycles per second (Hz) by the scaling $f_i = \omega_i/(2\pi)$.

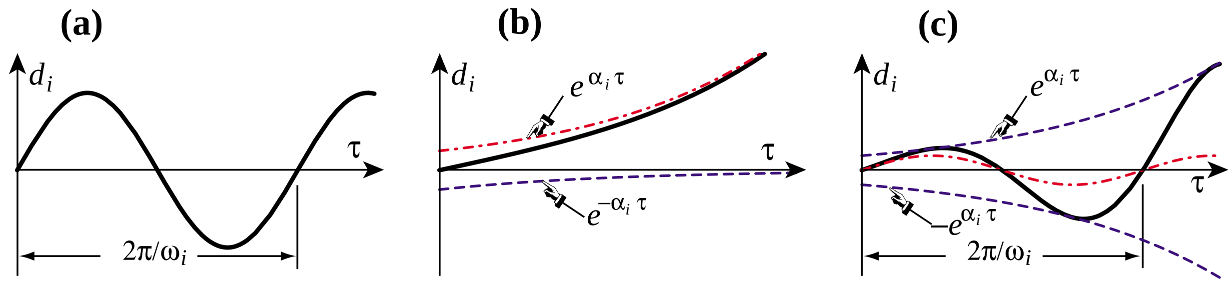

FIGURE 37.2. Depiction of 3 kinds of modal motion. (a): harmonic oscillatory motion for negative real p_i^2; equivalently, $p_i = \pm j\omega_i$ where ω_i is the circular frequency. (b): exponentially growing aperiodic motion for real $p_i^2 = \alpha_i^2$, $p_i = \pm\alpha_i$; transition to this kind of instability is called *divergence*. (c): periodic, exponentially growing motion for complex p_i^2 with $p_i = +\alpha_i \pm j\omega_i$, in which $\alpha_i \neq 0$; transition to this kind of instability is called *flutter*.

§37.4.1. Negative Real Root: Harmonic Oscillations

If p_i^2 is negative real, the square roots p_i are purely imaginary:

$$p_i = \pm j\omega_i, \qquad d(\tau) = \sum d_i(\tau), \quad d_i(\tau) = A_i \cos\omega_i\tau + B_i \sin\omega_i\tau. \tag{37.13}$$

The amplitudes A_i and B_i are determined by initial conditions. The motion d_i associated with $\pm j\omega_i$ is harmonic and bounded, as illustrated in Figure 37.2(a). The associated circular frequency is ω_i in rad/sec, the cyclic frequency is $f_i = \omega_i/(2\pi)$ in Hz, and the period is $T_i = 2\pi/\omega_i = 1/f_i$. The system is dynamically stable for that individual eigenvalue.

If *all* eigenvalues are negative real and distinct, the system is dynamically stable because any superposition of harmonic motions of different periods is also a harmonic (bounded) motion, as long as the number of DOF is finite. If two or more eigenvalues coalesce the analysis becomes more complicated because of the appearance of secular terms, but remains stable. Those effects can be studied in more detail in treatises in mechanical vibrations.

§37.4.2. Positive Real Root: Divergence

If p_i^2 is positive real,

$$p_i = \pm\alpha_i. \tag{37.14}$$

The $+\alpha_i$ square root will give rise to an *aperiodic*, exponentially growing motion. The other root will give rise to an exponentially decaying motion. When the two solutions are combined the exponentially growing one will dominate for sufficiently large τ, as pictured in Figure 37.2(b). The system is exponentially unstable.

As noted above p_i^2 is generally a function of λ. The transition from stability (in which all roots are negative real) to this type of instability necessarily occurs when a finite eigenvalue $p_i^2(\lambda)$, moving from left to right as λ varies, passes through the origin $p^2 = 0$ of the p^2 complex plane. This type of instability is called *divergence*. (The transition may occasionally occur at infinity frequency, as noted in Remark 37.6.)

§37.4.3. Complex Root: Flutter

If p_i^2 is complex, solutions of the eigenproblem (37.4) occur in *conjugate pairs* because both matrices **M** and **K** are real, as previously noted. Consequently, if $p_i^2 = (\alpha_i^2 - \omega_i^2) + j\,(2\alpha_i\omega_i)$ is a complex eigenvalue so is its conjugate $\overline{(p_i^2)} = (\alpha_i^2 - \omega_i^2) - j\,(2\alpha_i\omega_i)$. On taking the square root of this pair we find *four* characteristic exponents

$$\pm\alpha_i \pm j\omega_i. \tag{37.15}$$

Two of these square roots will have positive real parts $+\alpha_i$. For sufficiently large τ they will eventually dominate the other pair, yielding exponentially growing oscillations; see Figure 37.2(c). This is generally called *periodic exponential instability*, or *oscillatory instability*. In aero- and hydroelasticity, this is known as *flutter instability* or simply *flutter*.

If the system is initially stable (i.e., all roots are negative real) then transition to this type of instability occurs when at a certain value of λ *two real roots coalesce* on the real axis and "branch out" into the complex p^2 plane.

Remark 37.3. Frequency coalescence is necessary but not sufficient for flutter. It is possible for two frequencies to pass by each other "like ships crossing in the night" without merging. This happens if there is no mechanism by which the two associated eigenmodes can exchange energy.

Remark 37.4. The fact that all characteristic motions are either harmonic or exponentially growing is a consequence of the neglect of *damping* in setting up the stability problem. As noted in Remark 37.2, the presence of damping or, in general, dissipative forces, introduces additional mathematical complications that will not be elaborated upon here. Suffices to say that the addition of damping to a *conservative* system has always a *stabilizing* effect (Kelvin's theorem). For non-conservative systems, the preceding statement is no longer true, and indeed several counterexamples involving *destabilizing damping* have been constructed over the past 40 years. In spite of this the effect is not often observed in practice.

Remark 37.5. The occurrence of flutter requires the coalescence of *two* natural frequencies. Consequently, flutter cannot occur in systems with one degree of freedom ("it takes two to flutter"). The physical interpretation of the flutter phenomenon is that one vibration mode absorbs energy and feeds it into another. This transference or "energy resonance" becomes possible when the two modes have the same frequency.

Remark 37.6. Sometimes the flutter-triggering branching occurs at an infinite frequency. This happens in some load-follower problems. Such occurrence is called *flutter at infinity.* It may happen as a surprise that transition from stable to unstable does not happen at a critical point, but that theorem (due to Kelvin) is only valid for conservative systems. Occassionally the transition from stable to divergence also occurs at infinite frequence, as in the example of **?**. In that case we speak of *divergence at infinity*. The distinction between flutter and divergence ceases to be important in that case.

§37.4.4. Stable and Unstable Regions in the Complex Plane

From the preceding study it follows that the only stable region in the complex p^2-plane is the negative real axis:

$$\Re(p^2) < 0, \quad \Im(p^2) = 0. \tag{37.16}$$

The rest of the p^2 complex plane is unstable. See Figure 37.3(a).

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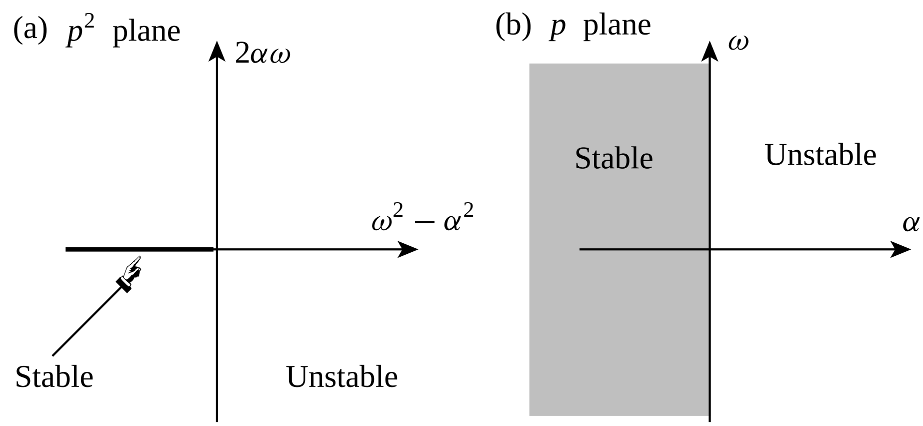

FIGURE 37.3. Stable and unstable regions in (a) the complex p^2 plane, (b) the complex p plane. For the latter the stable region is the left-half plane $\alpha = \Re(p) \leq 0$. For (a) it is the negative real axis.

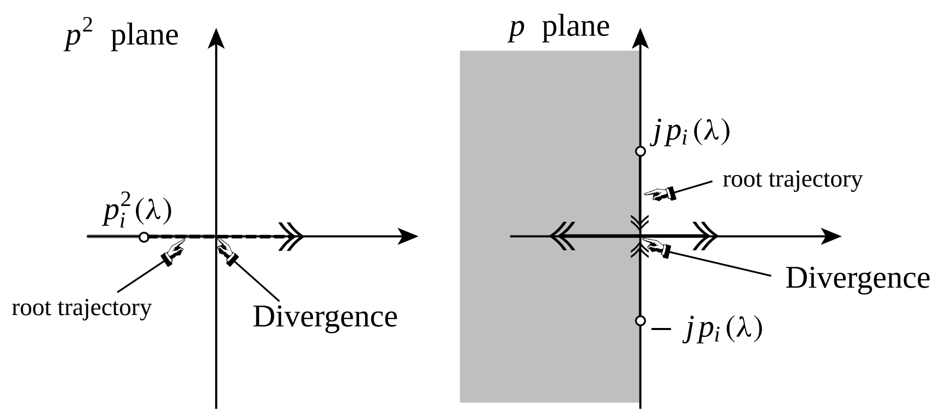

FIGURE 37.4. Root locus plots on the complex p^2 and p planes for divergence instability.

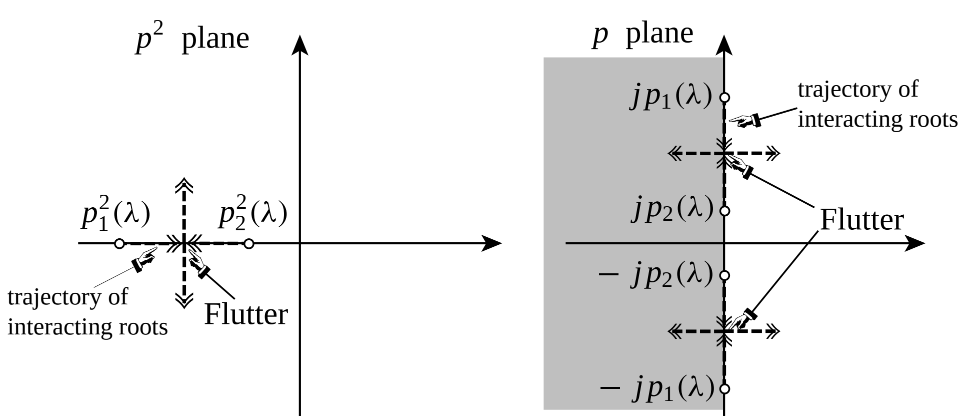

FIGURE 37.5. Root locus plots on the complex p^2 and p planes for divergence instability.

On the complex p-plane, the stable region is the left-hand plane

$$\alpha = \Re(p) \leq 0. \tag{37.17}$$

which includes the imaginary axis $\alpha = 0$ as stability boundary. The right-hand p-plane $\alpha > 0$ is unstable. See Figure 37.3(b).

§37.5. Graphical Representations

Two graphical representations that are often used in the literature on dynamic stability (and a related subject: active control systems) are described in the following subsections.

§37.5.1. Root Locus Plots

Graphical representations of the "trajectories" of the eigenvalues $p_i(\lambda)$ as λ is varied on the complex p^2 or p planes are valuable insofar as enhancing the understanding of the differences between divergence and flutter. These are called *root locus plots*,[6] and are illustrated in Figures 37.4 and 37.5.

Figure 37.4 illustrates loss of stability by divergence. As λ is varied, eigenvalue p_i^2 passes from the left-hand plane to the right-hand plane through the origin $p^2 = 0$. Stability loss occurs at the λ for which p_i^2 vanishes. The right-hand diagram depicts the same phenomenon on the p plane, for the root pair $\pm p_i$.

Figure 37.5 illustrates loss of stability by flutter. As λ is varied, two interacting eigenvalues, labeled as p_1^2 and p_2^2, coalesce on the negative real axis of the p^2 plane and branch out into the unstable region. The right-hand diagram depicts the same phenomenon on the p plane for the interacting roots, which appears in complex-conjugate pairs.

§37.5.2. Amplitude Plots

Another commonly used visualization technique is the *characteristic root amplitude* or simply *root amplitude* plots. These plots show the magnitude of $p_i(\lambda)$, that is $|p_i(\lambda)|$ on the vertical axis against λ on the horizontal axis. If the eigenvalue is real, $|p_i|$ is simply its absolute value whereas if it is complex $|p_i|$ is its modulus.

This graphical representation enjoys the following advantages: (a) the critical value of λ is displayed more precisely than with a locus or trajectory plot, (b) all related square roots such as $\pm\alpha_i \pm \omega_i$ "collapse" into a single value, and (c) the variation of several important roots (for several values of i) may be shown without cluttering the picture.

Figures 37.6(a) and 37.6(b) illustrate typical root-amplitude plots for loss of stability by divergence and flutter, respectively.

Variations of this technique are used in some examples of Chapter 35.

[6] The word *root* in root-locus is used as abbreviation for characteristic root or eigenvalue.

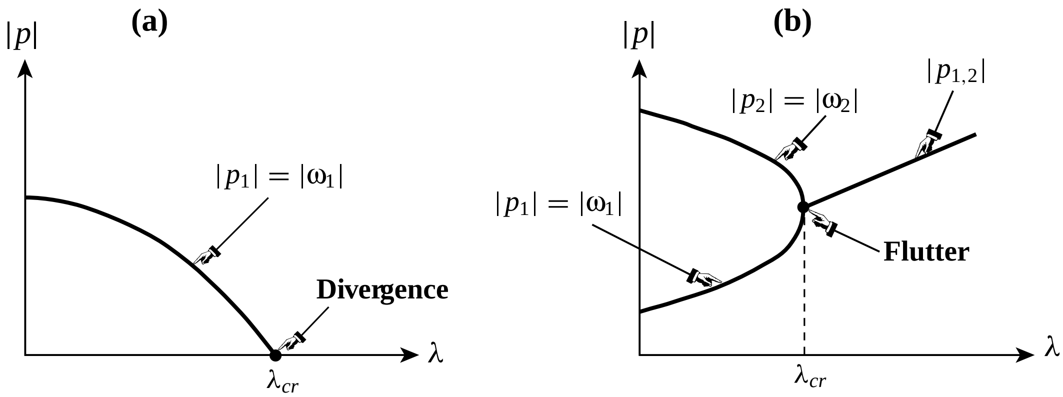

FIGURE 37.6. Root amplitude plots illustrating loss of stability by (a) divergence and (b) flutter, at $\lambda = \lambda_{cr}$.

§37.6. Regression to Zero Frequency and Static Tests

The stability loss by *divergence at zero* occurs when an eigenvalue p_i vanishes. Because $\omega_i = 0$ if $p_i = 0$, this is equivalent to a zero-frequency test on the eigenproblem

$$(-\omega_i^2 \mathbf{M} + \mathbf{K})\,\mathbf{z}_i = \mathbf{0}. \tag{37.18}$$

But if $\omega_i = 0$ and $\mathbf{M}$ is positive definite, which we assume, then $\mathbf{K}$ must be singular. Therefore we can regress to the static criterion or singular tangent stiffness test

$$\det\,\mathbf{K}(\lambda) = 0, \tag{37.19}$$

which allows us to *discard the mass matrix*. This regression may be useful if one is solving a series of closely related problems, for example during the design of a structure which is known *a priori* to become unstable by divergence. It should be noted, however, that the tangent stiffness matrix $\mathbf{K}$ for nonconservative systems is generally unsymmetric, whence the test for singularity must take account of that fact. For example, the usual pivot test on a $\mathbf{K} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T$ symmetric factorization must be replaced by a similar one on the more general $\mathbf{L}\,\mathbf{U}$ factorization.

The next Chapter goes over some classical examples of application of the linearized approach to dynamic stability. All of them are done analytically. Some examples are compared with a simple FEM discretization.

If divergence at infinity occurs, the dynamic criterion must be used, since there is no equivalent result within the singular-stiffness criteron.

Notes and Bibliography

Compared with the huge literature pile on classical structural stability, the treatment of dynamic stability in textbooks is sparse. One reason is that practical applications are largely confined to specialized engineering systems, such as suspension bridges, aircraft, rockets, turbines, and active control devices.

By far the best general description is the introductory chapter of Bolotin [101]. Despite its age, it has lost none of its freshness and physical insight. The Russian translation, edited by a well known expert in the area (G. Herrmann) is excellent, although as can be expected the numerical methods are way out of date. Bolotin has another book [102], which focuses on time-dependent and parametric excitation. In this one the exposition is flat, the material dated and the translation mediocre. Hard to believe it is the same author.

The second edition of Timoshenko and Gere [775], reprinted by Dover, covers dynamic stability briefly, through a couple of worked out examples; the numerical treatment is missing. The treatment in [65] is also brief, and entirely analytical. The best informal treatment appears in [570], but that book is hard to find nowadays. A recent book by Xie [856] treats more advanced topics, such as stochastic stability, ergodic excitation and Lyapunov exponents, and is a good source for reference material since 1960.