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Dynamic Stability: Formulation

Dynamic Stability Analysis

Required when system is nonconservative, either internally or externally.

General procedure:

At equilibrium configuration, set up equations of motion (EOM)

Apply a kinematically admissible perturbation (e.g. an initial displacement or velocity) as initial condition (IC)

Solve for dynamic response in real time

Check whether response stays bounded or unbounded (e.g. grows without limit)

Practical Difficulties With General Procedure

Requires mass and damping information to set up EOM

At each nonlinear equilibrium configuration we need to explore all admissible IC - time consuming

Nonlinear dynamic response computations are expensive. For complex models they require supercomputers

In Summary

A full systematic analysis is rarely justified, unless there are special circumstances, for example post-accident investigation for important litigation.

(In those cases the initial configuration & IC are known a priori)

One example: the post-mortem simulation of the World Towers collapse on 9/11/01. Litigation of owners with the insurance companies still proceeding. Full scale FEM simulations remain under wraps.

Simplifications

Linearize EOM: constant matrices and zero external force

Ignore damping (keep only mass matrix)

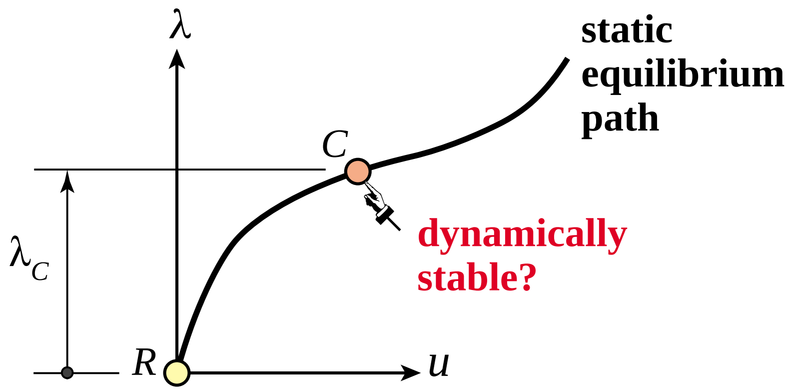
Separate spacetime variables, assuming exponential response in time to arbitrary IC

Analysis of roots of the steady-state response leads to a frequency eigenproblem, thus getting rid of IC and time

Study the eigenroots location (on the complex plane) as the control parameter is varied.

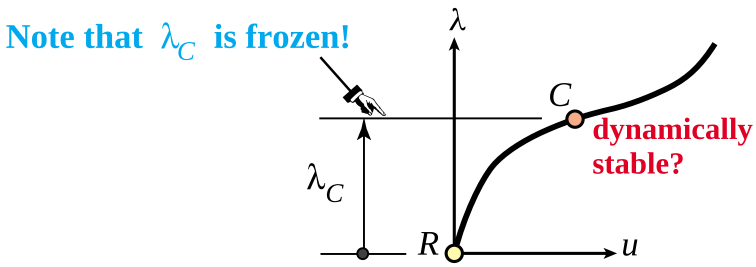
Information For Linearized Dynamic Stability Analysis

We would like to investigate the dynamic stability of a static equilibrium configuration C obtained in a response analysis :



using linearization. What do we need? Just two matrices: K & M - one more than for static stability

Information Needed For Linearized Dynamic Stability Analysis



K : tangent stiffness matrix evaluated at *C*
Important: must include all components:

material + geometric + load (conservative)
+ load (nonconservative)

M: mass matrix evaluated at *C*

Clarifying Questions

Should M be the actual mass matrix, or would a fictitious one do?

Depends on type of instability: need actual M for flutter
For divergence a fictitious M might be OK
To be safe, use the real thing

Where is damping?

It is usually safe to leave it out
That simplifies things, especially the eigenproblem

How about time-dependent forces?

Set them to zero
Only IC are applied as perturbation, and those will disappear on setting up the frequency eigenproblem

Notation Used for Perturbed Equations of Motion (EOM)

τ **will denote real time (recall that** t **is reserved for pseudotime)**
Perturbations are conventionally applied at $\tau = 0$

d **state vector perturbation** $\mathbf{d} = \mathbf{u}(\tau) - \mathbf{u}(0)$**, in which** $\mathbf{u}(0)$
is the state at the configuration being investigated
These will be considered infinitesimal

λ **will denote the frozen value of the control parameter**
at the configuration being investigated

$\dot{(\;)}$ **superposed dot denotes derivative wrt real time** τ **; for example**
$\ddot{\mathbf{d}}$ **is the real-time acceleration of the state perturbation**

Linearized Equations of Motion (EOM)

Dynamic EOM:

$$\mathbf{M}\,\ddot{\mathbf{d}} + \mathbf{K}\,\mathbf{d} = \mathbf{0}$$

where the explicit dependence of d(τ) on τ is usually omitted

Since the perturbed displacements d(τ) are assumed infinitesimal both M and K are kept constant

Perturbations at $\tau = 0$ are applied via initial conditions (IC):

$$\mathbf{d}(0) = \mathbf{d}_0 \qquad \dot{\mathbf{d}}(0) = \mathbf{v}_0$$

Modal Decomposition and the Complex Frequency (a.k.a. Characteristic) Eigenproblem

For constant M and K the solution of the dynamic EOM can be conveniently expressed in terms of the modal decomposition associated with the complex frequency eigenproblem

$$\boxed{(p_i^2\,\mathbf{M} + \mathbf{K})\;\mathbf{z}_i = \mathbf{0}} \qquad (*)$$

in which p_i **are eigenvalues called the complex frequencies, while** $\mathbf{z}_i$ **are the corresponding eigenvectors**

Math terminology: in applied math, (*) is called the characteristic eigensystem of $\mathbf{M}\,\ddot{\mathbf{d}} + \mathbf{K}\,\mathbf{d} = \mathbf{0}$. **Its eigenvalues are the characteristic roots. The determinant of the matrix in parenthesis, equated to zero, is called the characteristic equation**

Complex Frequency Eigenproblem Properties

Because K is real but generally unsymmetric (otherwise there would no reason to do the dynamic stability analysis) the eigenvalues p_i^2 can be complex numbers. Furthermore, the entries of the eigenvectors $\mathbf{z}_i$ can be also complex.

But since K and M are real and M is symmetric P.D., the following properties hold:

- **Eigenvalues come in complex conjugate pairs. That is, if $p_i = \alpha_i + j\,\omega_i$ is an eigenvalue, so is $\overline{p}_i = \alpha_i - j\,\omega_i$ where $j = \sqrt{-1}$ is the imaginary unit**

- **Eigenvectors corresponding to complex conjugate eigenvalues are also conjugate. That is, if the eigenvector corresponding to p_i is $\mathbf{z}_i$, that corresponding to $\overline{p}_i$ is $\overline{\mathbf{z}}_i$ (of course, this assumes uniform eigenvector normalization)**

Connection to the Vibration Eigenproblem

The complex frequency eigenproblem of the previous slide may be viewed as a generalization of the better known vibration eigenproblem

$$(-\omega_i^2\ \mathbf{M} + \mathbf{K})\,\phi_i = \mathbf{0}$$

in which M and K are symmetric, M is P.D. and K is N.N.D.
Under such conditions the squared vibration frequencies ω_i^2 **are real and nonnegative, and the vibration eigenmodes** ϕ_i **are real.**
If the two eigenproblems overlap, they are linked by

$$p_i^2 = -\omega_i^2 \qquad \phi_i = \mathbf{z}_i$$

That is, $p_i = \pm\, j\, \omega_i$ **, where** $j = \sqrt{-1}$ **denotes the imaginary unit.**
For example, if $\omega^2 = 4,\ \ p = \pm\, 2j$

General Solution

The general solution of $\mathbf{M}\,\ddot{\mathbf{d}} + \mathbf{K}\,\mathbf{d} = 0$ in terms of the eigenmodes is a sum of exponentials:

$$\mathbf{d}(\tau) = \sum_i \mathbf{d}_i(\tau) = \sum_i C_i\, \mathbf{z}_i\, e^{p_i \tau}$$

where the sum extends to $i = 1, \ldots 2N$, N **being the number of degrees of freedom (DOF).**

The coefficients C_i **may be found by projecting the IC on the eigenvectors** $\mathbf{z}_i$ **if so derired. But we shall see that the key information as regards dynamic stability is provided by the eigenvalues** p_i **.**
In that respect, those coefficients are irrelevant.

Notice that several quantities under the sum: C_i **,** $\mathbf{z}_i$ **,** p_i **, can be complex. But d(τ) is of course real. How come? The complex conjugacy property saves the day.**

Stability By Looking At Modes

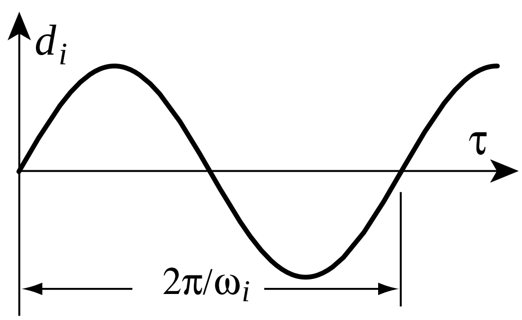
Since the dynamic response $\mathbf{d}(\tau)$ is the sum of modal responses $\mathbf{d}_i(\tau)$, to assess stability it is sufficient to look at each mode in turn and thus arrive to one of these conclusions:

- **If each modal response is stable (= bounded in time) so is their sum and the system is dynamically stable**
- **If at least one modal response is unstable (= unbounded in time) so is the sum, and the system is dynamically unstable**
- **A third possibility is transition from stable to unstable. The system is then at a critical state known as neutral dynamic stability (unlike static stability, this does not necessarily happens at a critical point, as defined in Chapter 5)**

We therefore reduce the stability problem to the examination of individual modal responses. These are completely determined by the associated eigenvalues, as studied next.

Case 1: p_i^2 Negative Real

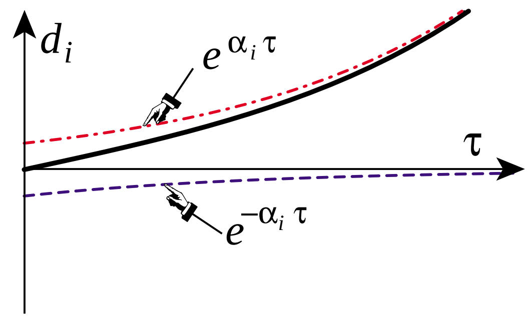
If p_i^2 is negative real, its two square roots are purely imaginary. The corresponding modal contribution from that root pair, using Euler's formula, comes out to be a sine-cosine combination, which is real, oscillatory and bounded:



This case overlaps with a vibration frequency ω_i so that $p_i^2 = -\,\omega_i^2$
This modal component is stable

Case 2: p_i^2 Positive Real

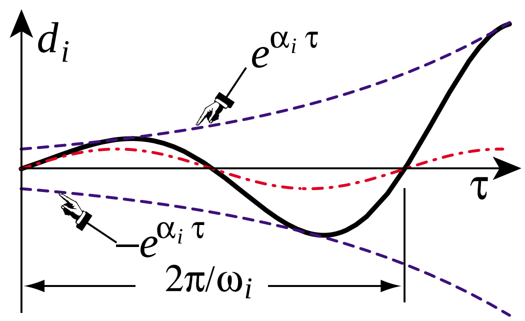
If p_i^2 is positive real, say α_i^2, its two square roots are real, one positive and one negative. The real root $+\alpha_i$ produces an exponentially growing contribution that eventually overwhelms the other one:



This modal component will produce divergence instability

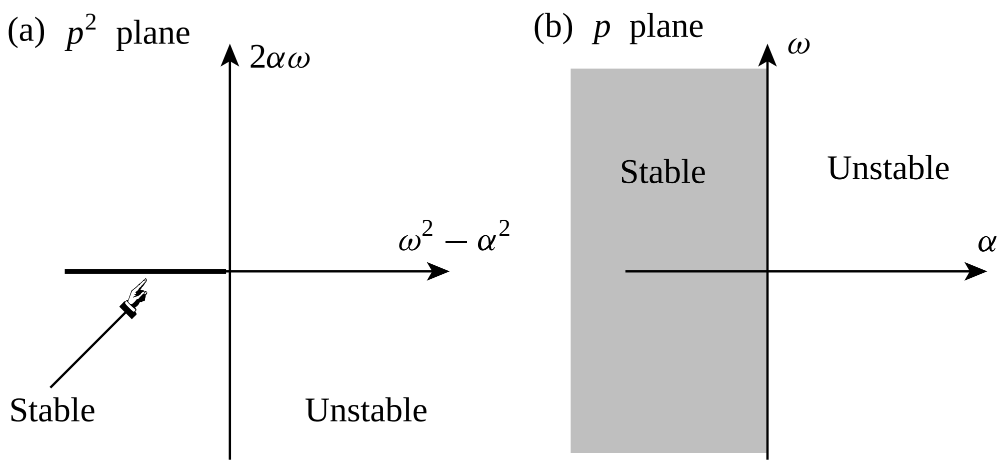
Case 3: p_i^2 Complex

If p_i^2 is complex, its complex conjugate $\bar{p}_i^2$ is also an eigenvalue. The square roots of this pair collectively yield four eigenvalues of the form $\pm\,\alpha_i\,\pm j\,\omega_i$, in which α_i and ω_i are positive. Two roots out of the four will have positive real components responsible for growing oscillations:

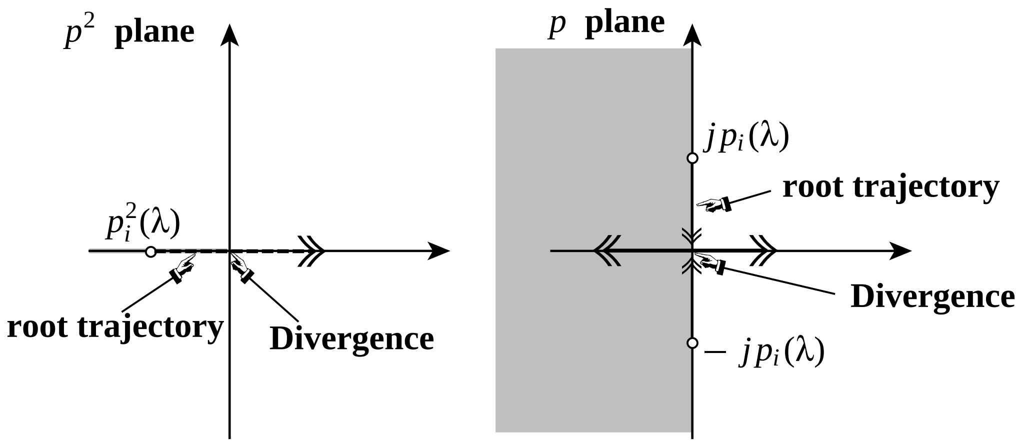


This modal component will produce flutter instability

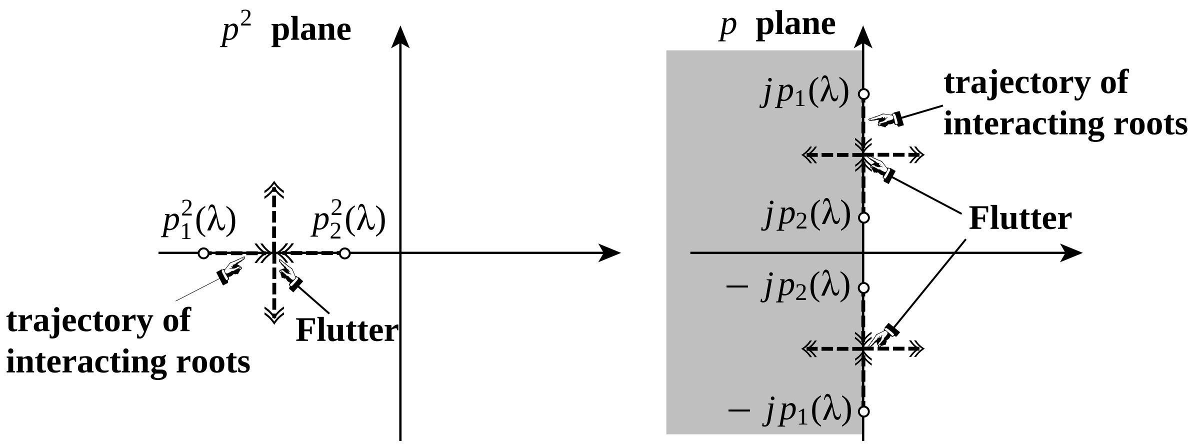
Graphical Representation on the Complex p^2 or p Planes



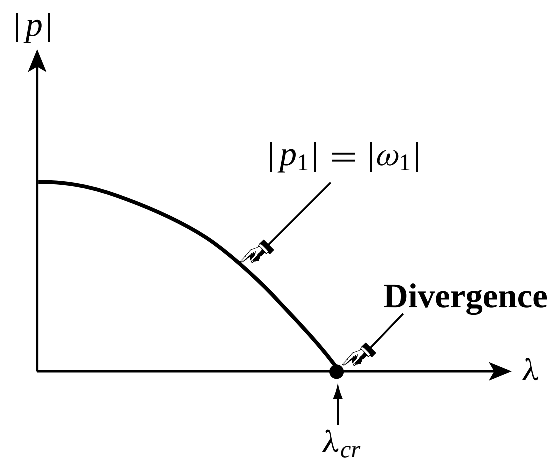
Root Trajectory Representation of Divergence Instability on Complex Plane



Root Trajectory Representation of Flutter Instability On Complex Plane

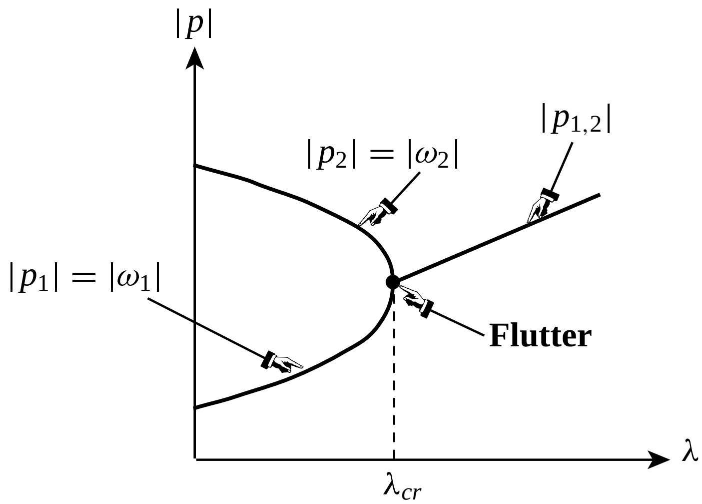


Amplitude-Plot Representation of Divergence Instability



Assume root #1 passes through plane origin producing divergence

Amplitude-Plot Representation of Flutter Instability



Assume roots # 1 and #2 coalesce producing flutter