35

Nonconservative Loading: Overview

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§35.1. Introduction

In Chapter 6 a mechanical system was called *conservative* when both external *and* internal forces are derivable from a potential. In this course we consider only elastic systems, whence the internal forces are always derivable from an internal (a.k.a. strain) energy potential U. Therefore the conservative-or-not label depends on the nature of the *external loads*. *Conservative* applied forces $\mathbf{f}$ may be derived from the external work potential W by differentiating with respect to the state variables:

$$\mathbf{f} = \frac{\partial W}{\partial \mathbf{u}}. \tag{35.1}$$

Nonconservative forces, on the other hand, are not expressable as (35.1). They have to be worked out directly at the force (residual) level, although the Principle of Virtual Work (PVW) continues to be broadly applicable to get consistent force lumping.

The present Chapter provides a general overview of how nonconservative forces are handled. The main result is they contribute an unsymmetric component, called *load stiffness*, to the tangent stiffness matrix. A consequence of the symmetry loss is that stability analysis requires a dynamic criterion, which is covered in Chapter 37. In this and next Chapter examples of both force types are given for several one-and two-dimensional elements moving in 2D and 3D space. The Total Lagrangian (TL) kinematic description will be favored in the examples.

§35.2. Sources

Some sources of nonconservative forces in various branches of engineering are:

1. **Aerodynamic** forces (aerospace, civil); **hydrodynamic** forces (mechanical, marine, chemical); aircraft and rocket **propulsion** forces (aerospace); **frictional** forces (aerospace, mechanical, civil); **electromagnetic** forces (electrical).
2. Gyroscopic forces (aerospace, electrical).[1]
3. Active control systems (aerospace, electrical, mechanical).
4. Compressive loads carried by the human lumbar spine (bioengineering).[2]

In this and the next Chapter we consider mainly aero- and hydrodynamic forces, which are due to relative fluid motion, in the prototype examples.

§35.3. Three Examples

The following examples are designed to illustrate differences between conservative and nonconservative loads, as well as the appearance of the load stiffness matrix in the latter case. The Total Lagrangian (TL) kinematic description is used in all examples.

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[1] Gyroscopic forces are “pseudo-conservative” in the sense thay they do no work and hence do not have a potential. This topic is beyond our scope.

[2] In vivo experiments shows that the human spine buckles at 80-100N. But while standing and walking the compressive force may reach 1000N, and nothing happens. The discrepancy has been attributed, among various factors, to the nonconservative (“follower”) nature of the loading.

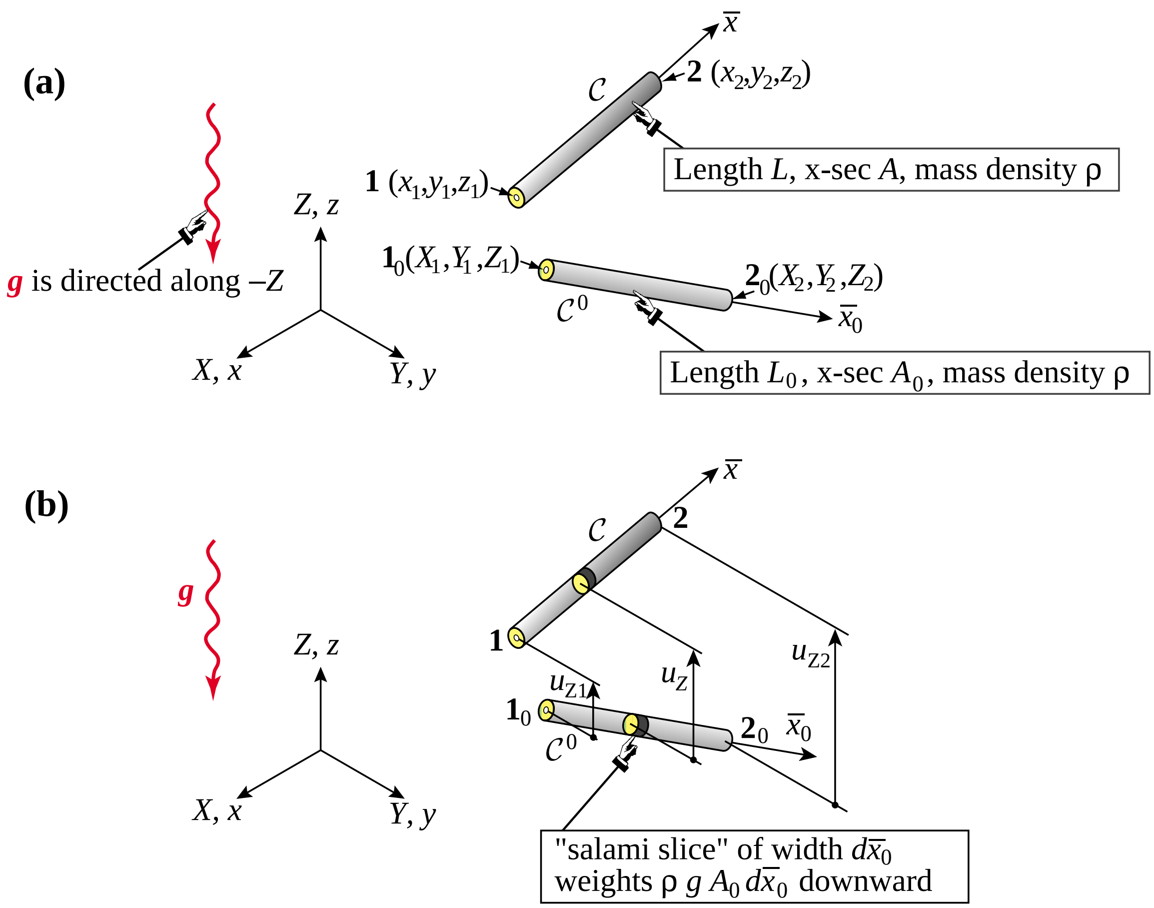

FIGURE 35.1. TL bar element moving in space under gravity field g (shown in red): (a) initial (reference) and current configurations; (b) external potential calculation.

§35.3.1. Bar Moving in 3D Under Gravity

Consider a two-node, prismatic, straight bar element moving in three dimensional space. The element is immersed in a gravity field of constant strength g acting along the global $-Z$ axis, as pictured in Figure 35.1(a). The bar has reference (initial) length L_0, reference (initial) area A_0 and uniform mass density ρ. The local (element) coordinate systems are labeled as follows:

$$\bar{x}_0, \bar{y}_0, \bar{z}_0 \qquad \text{in the reference configuration } \mathcal{C}^0$$

$$\bar{x}, \bar{y}, \bar{z} \qquad \text{in the current configuration } \mathcal{C}$$

In both configurations the origin of local coordinates is at node 1. Note that the direction of the local $\bar{y}_0$, $\bar{z}_0$, $\bar{y}$, $\bar{z}$ axes is irrelevant. (This distinction between local coordinate systems is introduced here as it becomes necessary in the next two examples.) The element DOF are ordered $\mathbf{u} = [\, u_{X1} \;\; u_{Y1} \;\; u_{Z1} \;\; u_{X2} \;\; u_{Y2} \;\; u_{Z2} \,]^T$. Take a differential element (“bar salami slice”) of length $d\bar{x}_0$ in $\mathcal{C}^0$. This moves to a corresponding position in $\mathcal{C}$, with a vertical displacement of u_Z with respect to $\mathcal{C}^0$, as illustrated in Figure 35.1(b). The work spent by performing this motion is

$$dW = -\rho g A_0 u_Z(\bar{x}_0)\, d\bar{x}_0 \tag{35.2}$$

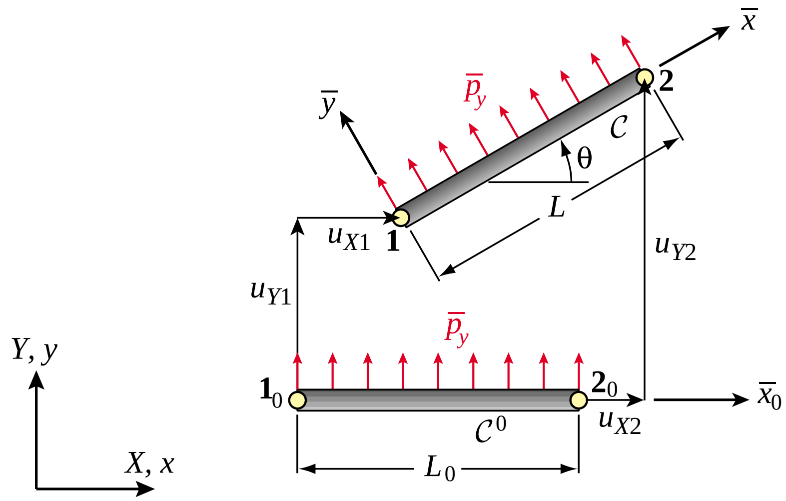

FIGURE 35.2. Two-node bar element moving in 2D under constant "follower" pressure $\bar{p}_y$, shown in red.

The external work potential of the whole element is obtained by linearly interpolating the $Z \equiv z$ motion:

$$u_Z = (1-\zeta)\, u_{Z1} + \zeta\, u_{Z2}, \tag{35.3}$$

in which $\zeta = \bar{x}_0/L_0$ is a natural coordinate. Integrating over the bar length yields

$$W = -\int_0^{L_0} \rho g A_0\, u_Z\, d\bar{x}_0 = -\rho\, A_0\, q \int_0^1 [\,1-\zeta \quad \zeta\,] \begin{bmatrix} u_{Z1} \\ u_{Z2} \end{bmatrix} L_0\, d\zeta = -\rho g A_0 L_0\, \tfrac{1}{2}(u_{Z1} + u_{Z2}), \tag{35.4}$$

to which an arbitrary constant can be added. As usual in the TL kinematic description, all quantities are referred to $\mathcal{C}^0$. It follows that the external force vector for the element is

$$\mathbf{f}_g = \frac{\partial W}{\partial \mathbf{u}} = \begin{bmatrix} \partial W/\partial u_{X1} \\ \partial W/\partial u_{Y1} \\ \partial W/\partial u_{Z1} \\ \partial W/\partial u_{X2} \\ \partial W/\partial u_{Y2} \\ \partial W/\partial u_{Z2} \end{bmatrix} = -\tfrac{1}{2}\rho A_0 L_0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{35.5}$$

This could also be derived through elementary statics using, for example, element-by-element force lumping. Note that vector $\mathbf{f}_g$ is *independent of the current configuration*. This is a distinguishing feature of external work potentials that *depend linearly on the displacements*, such as (35.4).

§35.3.2. Bar in 2D Under Follower Load

To illustrate the concept of *load stiffness* with a minimum of mathematics, let us consider first a two-dimensional specialization. The 2-node TL bar element originally lies along the X axis in the reference configuration $\mathcal{C}^0$ so $\bar{x}_0 \parallel X$. See Figure 35.2. The bar moves in the (X, Y) plane to configuration $\mathcal{C}$, which forms an angle θ with X. Bar lengths are L_0 and L, respectively. The element node displacements are collected in the 4-vector

$$\mathbf{u} = [\,x_1 \quad y_1 \quad x_2 \quad y_2\,]^T - [\,X_1 \quad Y_1 \quad X_2 \quad Y_2\,]^T = [\,u_{X1} \quad u_{Y1} \quad u_{X2} \quad u_{Y2}\,]^T. \tag{35.6}$$

The bar is under a *constant* distributed load $\bar{p}_y$ (force per unit length) that remains *normal* to the element as it displaces, as shown in Figure 35.2. This kind of applied force is called a *follower load* in the literature.[3] From elementary statics the external force vector in 2D is obviously

$$\mathbf{f}_p = \tfrac{1}{2}\bar{p}_y\, L \begin{bmatrix} -\sin\theta \\ \cos\theta \\ -\sin\theta \\ \cos\theta \end{bmatrix} \tag{35.7}$$

From inspection

$$\cos\theta = \frac{L_0 + u_{X21}}{L}, \quad \sin\theta = \frac{u_{Y21}}{L}, \tag{35.8}$$

in which $u_{X21} = u_{X2} - u_{X1}$ and $u_{Y21} = u_{Y2} - u_{Y1}$. Consequently

$$\mathbf{f}_p = \tfrac{1}{2}\bar{p}_y\, L \begin{bmatrix} -u_{Y21}/L \\ (L_0 + u_{X21})/L \\ -u_{Y21}/L \\ (L_0 + u_{X21})/L \end{bmatrix} = \tfrac{1}{2}\bar{p}_y \begin{bmatrix} -u_{Y21} \\ L_0 + u_{X21} \\ -u_{Y21} \\ L_0 + u_{X21} \end{bmatrix}. \tag{35.9}$$

Take now the partial with respect to $\mathbf{u}$ of the negative of this load vector. The result is a matrix with dimensions of stiffness (i.e., force over length), which is denoted by $\mathbf{K}_L$:

$$\mathbf{K}_L \stackrel{\text{def}}{=} -\frac{\partial \mathbf{f}_p}{\partial \mathbf{u}} = \begin{bmatrix} \partial\mathbf{f}_p/\partial u_{X1} \\ \partial\mathbf{f}_p/\partial u_{Y1} \\ \partial\mathbf{f}_p/\partial u_{X2} \\ \partial\mathbf{f}_p/\partial u_{Y2} \end{bmatrix} = \tfrac{1}{2}\bar{p}_y \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}. \tag{35.10}$$

$\mathbf{K}_L$ is called a *load stiffness matrix*.[4] It arises from *displacement-dependent loads*.[5] We can see from this example that $\mathbf{K}_L$ is *unsymmetric*. A consequence of this fact is that $\mathbf{f}$ does not have a potential W that is a function of the node displacements.[6]

§35.3.3. Triangular Membrane Plate in 3D Under Follower Pressure

The last example considers a 3-node flat plate triangular element moving in 3D space, and subjected to a *constant* lateral pressure $\bar{p}_z$. The pressure is positive if it is directed along the positive normal defined below. See Figure 35.3(a). This element is in a membrane (plate stress) state and has no bending rigidity. This is a FEM model appropriate for very thin shells, for example an inflating balloon or a boat sail. A TL kinematic description is used. The plate is assumed to be materially homogenous. Consequently all FEM quantities introduced below are referred to the midsurface.

The three nodes are located at the midsurface corners. The node coordinates in the reference configuration $\mathcal{C}^0$ are $\{X_i, Y_i, Z_i\}$ ($i = 1, 2, 3$), the triangle area is A_0, and the midsurface normal

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[3] They are called *slave loads* in English translation of some Russian books, e.g. [570]. Such loads are often applied by fluids at rest or in motion. Aero- and hydrodynamic loading is studied in more detail in the next Chapter.

[4] Also called *follower stiffness matrix* in the literature when it is associated with follower forces.

[5] This source of nonlinearity was called *force B.C.* nonlinearity in Chapter 2.

[6] If $\mathbf{K}_L$ were symmetric we could do "reverse engineering" starting with (35.9), and integrating the associated variational expression $\delta W = \mathbf{f}_p\, \delta\mathbf{u}$ with respect to node displacements to find the external work potential W.

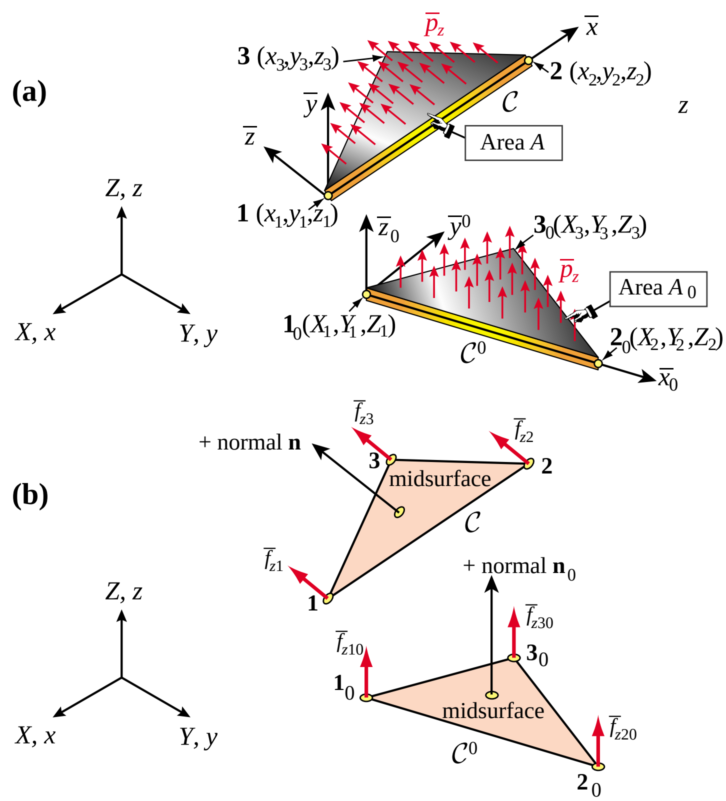

FIGURE 35.3. Triangular plate element moving in 3D space and subject to constant lateral pressure $\bar{p}_z$ (shown in red): (a) reference and current configurations; (b) nodal forces from pressure lumping.

is $\mathbf{n}_0$. The node coordinates in the current configuration $\mathcal{C}$ are $\{x_i, y_i, z_i\}$ (i=1,2,3), the triangle area is A, and the midsurface normal is $\mathbf{n}$. The positive sense of the normals is chosen so that the side circuit $1 \rightarrow 2 \rightarrow 3$ is traversed CCW when looking down from the normal vector tip; cf. Figure 35.3(b). The element nodal displacements are collected in the 9-vector

$$\mathbf{u} = [\,x_1 \quad y_1 \quad z_1 \quad \ldots \quad z_3\,]^T - [\,X_1 \quad Y_1 \quad Z_1 \quad \ldots \quad Z_3\,]^T = [\,u_{X1} \quad u_{Y1} \quad u_{Z1} \quad \ldots u_{Z3}\,]^T. \tag{35.11}$$

The local axes are chosen as pictured in Figure 35.3(a). In $\mathcal{C}$, $\bar{x}$ is directed along side $(1,2)$, with origin at corner 1; $\bar{z}$ is parallel to the normal $\mathbf{n}$ at 1; $\bar{y}$ is taken normal to both $\bar{x}$ and $\bar{z}$ forming a RHS frame. The local axes in $\mathcal{C}^0$: $\{\bar{x}_0, \bar{y}_0, \bar{z}_0\}$ are chosen similarly. As usual we will employ the abbreviations $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, $z_{ij} = z_i - z_j$, $X_{ij} = X_i - X_j$, $Y_{ij} = Y_i - Y_j$, and $Z_{ij} = Z_i - Z_j$ for node coordinate differences.

The analysis that follows is largely based on the treatment of a 4-node tetrahedron in [263, Ch. 9]. The *direction numbers*, or simply *directors*, of the normal $\mathbf{n}$ to the triangle at $\mathcal{C}$ (same as those of the $\bar{z}$ axis) with respect to the global frame are

$$a_n = y_{13}\,z_{21} - y_{12}\,z_{31}, \qquad b_n = x_{21}\,z_{13} - x_{31}\,z_{12}, \qquad c_n = x_{13}\,y_{21} - x_{12}\,y_{31}. \tag{35.12}$$

These have dimensions of length squared. The triangle area in $\mathcal{C}$ is given by

$$A = \tfrac{1}{2}\,S_n, \qquad \text{in which } S_n = +\sqrt{a_n^2 + b_n^2 + c_n^2} \tag{35.13}$$

The *direction cosines* $\{\alpha_n, \beta_n, \gamma_n\}$ of $\mathbf{n}$ with respect to the global frame are obtained by scaling the directors (35.12) to unit length:

$$\alpha_n = a_n/S_n, \quad \beta_n = b_n/S_n, \quad \gamma_n = c_n/S_n. \tag{35.14}$$

The total pressure force in $\mathcal{C}$ is obviously pressure $\times$ triangular area, or $\bar{p}_z\, A$, positive if along $+\mathbf{n}$. This is statically lumped into 3 equal corner node forces: $\bar{f}_{z1} = \bar{f}_{z2} = \bar{f}_{z3} = \frac{1}{3}\,\bar{f}_z\,\bar{p}_z\, A$, as pictured in Figure 35.3(b). The global frame components of these node forces are obtained by multiplying them by the direction cosines, and then using (35.14) and (35.12). Here are the details:

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \\ f_{x2} \\ f_{y2} \\ f_{z2} \\ f_{x3} \\ f_{y3} \\ f_{z3} \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\, A \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \alpha_n \\ \beta_n \\ \gamma_n \\ \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\, A \begin{bmatrix} a_n/S_n \\ b_n/S_n \\ c_n/S_n \\ a_n/S_n \\ b_n/S_n \\ c_n/S_n \\ a_n/S_n \\ b_n/S_n \\ c_n/S_n \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\, A \begin{bmatrix} a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \\ a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \\ a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \end{bmatrix} = \tfrac{1}{6}\,\bar{p}_z \begin{bmatrix} a_n \\ b_n \\ c_n \\ a_n \\ b_n \\ c_n \\ a_n \\ b_n \\ c_n \end{bmatrix}. \tag{35.15}$$

Note that A cancels out. As a consequence square roots and denominators disappear and each $\mathbf{f}$ entry is simply a quadratic polynomial in coordinates and displacements. To get the load stiffness it is necessary to express a, b, c in terms of the reference node coordinates and the node displacements. For example

$$x_{21} = X_{21} + u_{X21} = X_2 - X_1 + u_{X2} - u_{X1}, \quad \text{etc.} \tag{35.16}$$

This is followed by differentiation with respect to the components of $\mathbf{u}$. The derivations are posed as an Exercise.

§35.4. General Characterization of the Load Stiffness

Suppose that we have a one-control-parameter system with displacement-dependent loads. We will distinguish two cases: whether the loading is fully conservative (derives from an external potential), or not. A load stiffness matrix will appear in both cases, but methods used in the subsequent stability analysis will differ.

§35.4.1. Displacement-Dependent Conservative Loads

Consider a total potential energy of the form

$$\Pi(\mathbf{u}, \lambda) = U(\mathbf{u}) - W(\mathbf{u}, \lambda), \tag{35.17}$$

in which the external work potential $W = W(\mathbf{u}, \lambda)$ is taken to depend on the state vector $\mathbf{u}$ in a general fashion. Then

$$\mathbf{r} = \frac{\partial \Pi}{\partial \mathbf{u}} = \frac{\partial U}{\partial \mathbf{u}} - \frac{\partial W}{\partial \mathbf{u}} = \mathbf{p} - \mathbf{f}, \tag{35.18}$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}. \tag{35.19}$$

The partial $\partial\mathbf{p}/\partial\mathbf{u}$ gives $\mathbf{K}_M + \mathbf{K}_G$, the material plus geometric stiffness, as discussed in previous Chapters. The last term gives $\mathbf{K}_L$, the *conservative load stiffness*

$$\mathbf{K}_L = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = -\frac{\partial^2 W}{\partial \mathbf{u}^2} \tag{35.20}$$

This matrix is symmetric because it is the negated Hessian of $W(\mathbf{u}, \lambda)$ with respect to $\mathbf{u}$. Consequently

$$\boxed{\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_L.} \tag{35.21}$$

These three components of $\mathbf{K}$ are symmetric, and so is $\mathbf{K}$. Previous analysis methods apply; the only difference being that the tangent stiffness now splits into three components.

§35.4.2. Displacement-Dependent NonConservative Loads

Now consider a more general structural system subject to both conservative and *non-conservative* loads:

$$\boxed{\mathbf{r} = \mathbf{p} - \mathbf{f}_c - \mathbf{f}_n,} \tag{35.22}$$

Here $\mathbf{f}_c = \partial W/\partial\mathbf{u}$ whereas $\mathbf{f}_n$ collects external forces not derivable from a potential. Then

$$\boxed{\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_{Lc} + \mathbf{K}_{Ln}.} \tag{35.23}$$

The nonconservative load stiffness matrix, $\mathbf{K}_{Ln}$, is *unsymmetric*.

Remark 35.1. In practice one may derive the applied force $\mathbf{f}$ from statics, as in the examples of §35.3.2 and §35.3.3. The load stiffness $\mathbf{K}_L$ follows on taking partials with respect to the displacements in $\mathbf{u}$. If $\mathbf{K}_L$ is symmetric the applied load is conservative, and one may work backward to get the potential W. If the resulting stiffness is unsymmetric the load is nonconservative. The splitting of $\mathbf{K}_L$ into a symmetric matrix $\mathbf{K}_{Lc}$ and unsymmetric part $\mathbf{K}_{Ln}$ can be done in several ways. (If the unsymmetric part is required to be antisymmetric, however, the splitting is unique.)

Homework Exercises for Chapter 35

Nonconservative Loading: Overview

EXERCISE 35.1 [A:15] For the example treated in §35.3.2 (a bar moving in 2D under a follower force), assume now that $\bar{p}_y$ depends on the tilt angle θ so that

$$\bar{p}_y = \bar{p}\,\cos\theta. \tag{E35.1}$$

Obtain the associated node force vector $\mathbf{f}_p$ and the load stiffness $\mathbf{K}_L$.

Hint. For connections between tilt angle trig functions and node displacements, see (36.7).

EXERCISE 35.2 [A:20] Complete the example treated in §35.3.3 by getting the load stiffness $\mathbf{K}_L$. Check whether it is symmetric or unsymmetric.

Although the computations can be in principle done by hand, getting all $9 \times 9 = 81$ entries of $\mathbf{K}_L$ becomes tedious and error prone. Time can be saved by using a computer algebra system (CAS). For example the following *Mathematica* script may be used to carry out the derivations:

```
SpaceTrigDirectors[ncoor_]:=Module[{x1,y1,z1,x2,y2,z2,x3,y3,z3,a,b,c},
  {{x1,y1,z1},{x2,y2,z2},{x3,y3,z3}}=ncoor;
  x12=x1-x2; x21=-x12; y12=y1-y2; y21=-y12; z12=z1-z2; z21=-z12;
  x23=x2-x3; x32=-x23; y23=y2-y3; y32=-y23; z23=z2-z3; z32=-z23;
  x31=x3-x1; x13=-x31; y31=y3-y2; y13=-y31; z31=z3-z1; z13=-z31;
  a=y13*z21-y12*z31; b=x21*z13-x31*z12; c=x13*y21-x12*y31;
  Return[Simplify[{a,b,c}]]];
ClearAll[X1,Y1,Z1,X2,Y2,Z2,X3,Y3,Z3];
ClearAll[x1,y1,z1,x2,y2,z2,x3,y3,z3];
ClearAll[uX1,uX2,uX3,uY1,uY2,uY3,uZ1,uZ2,uZ3,pz]; pz=1;
rep={x1->X1+uX1,x2->X2+uX2,x3->X3+uX3, y1->Y1+uY1,y2->Y2+uY2,y3->Y3+uY3,
     z1->Z1+uZ1,z2->Z2+uZ2,z3->Z3+uZ3};
rep0={uX1->0,uY1->0,uZ1->0,uX2->0,uY2->0,uZ2->0,uX3->0,uY3->0,uZ3->0};
ncoor={{x1,y1,z1},{x2,y2,z2},{x3,y3,z3}};
{an,bn,cn}=SpaceTrigDirectors[ncoor];
Print["Directors of normal:",{an,bn,cn}];
{an,bn,cn}=Simplify[{an,bn,cn}/.rep];
Print["Directors upon rep:",{an,bn,cn}];
f=(pz/6)*{an,bn,cn,an,bn,cn,an,bn,cn}; f0=Simplify[f/.rep0];
Print["f=",f//MatrixForm]; Print["f0=",f0//MatrixForm];
KL=-{D[f,uX1],D[f,uY1],D[f,uZ1],D[f,uX2],D[f,uY2],D[f,uZ2],
     D[f,uX3],D[f,uY3],D[f,uZ3]}; KL=Simplify[KL];
Print["KL=",KL//MatrixForm];
Print["symm chk=",Simplify[Kl-Transpose[KL]]//MatrixForm];
KL0=Simplify[KL/.rep0]; Print["KL0=",KL0//MatrixForm];
```

Here symbol names ending with 0 (zero) pertain to $\mathcal{C}^0$.

EXERCISE 35.3 [A:15] Show that extending the example of §35.3.2 to 3D is impossible without extra assumptions on the follower load direction. *Hint*: uncertainty in specification of local axes.