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Nonconservative Loading: Overview

Nonconservative Loading

New class of problems:

External loads (or part of them) do not have a potential

Consequences:

Equilibrium equations must be set up at the residual level

Stiffness matrix acquires an unsymmetric component

Stability analysis must use a dynamic criterion
even if loads are static

New Instability Phenomena of Dynamic Type

Divergence: exponentially growing monotone motion

Flutter: exponentially growing oscillation

These will be analyzed in Chapters 37-38.

Chapter 34 explain on how to incorporate nonconservative forces into the governing equations
Chapter 35 focuses on aero- and hydrodynamic loads

The External Work Potential

Conservative applied forces may be derived from an external work potential W by differentiating it with respect to state variables:

$$\boxed{\mathbf{f} = \frac{\partial W}{\partial \mathbf{u}}}$$

Nonconservative forces are not expressable in this way. They have to be worked out directly at the force residual level.

Sources

Liquid or gas in motion:

Aerodynamic forces (aerospace, civil)

Hydrodynamic forces (mechanical, marine, chemical, petroleum)

Rocket & jet propulsion forces (aerospace)

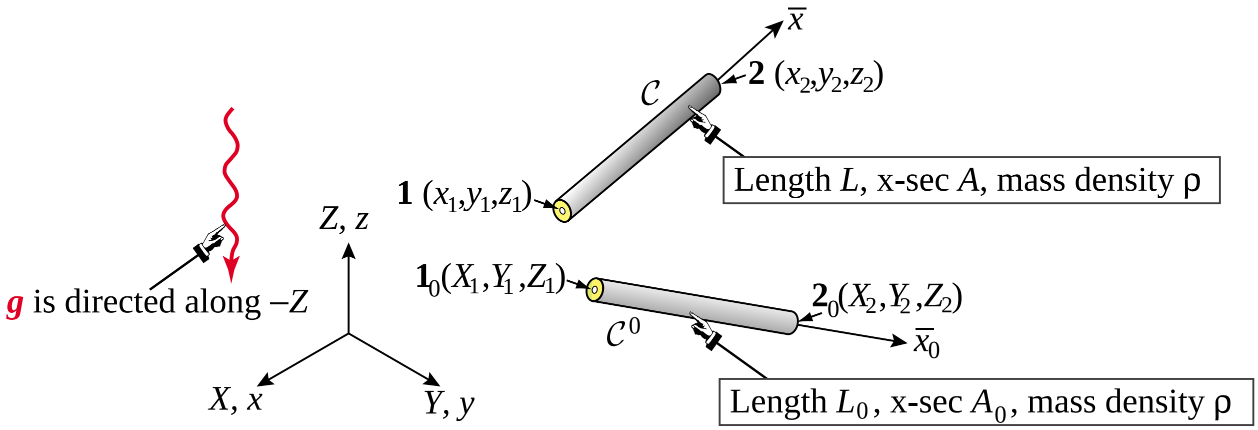
Frictional forces (aerospace, mechanical, civil)

Gyroscopic and rotating forces (aerospace, mechanical, electrical)

Active control systems (aerospace, electrical, mechanical)

Some bioengineering systems: e.g. human lumber spine

Potential Force Example: Gravity

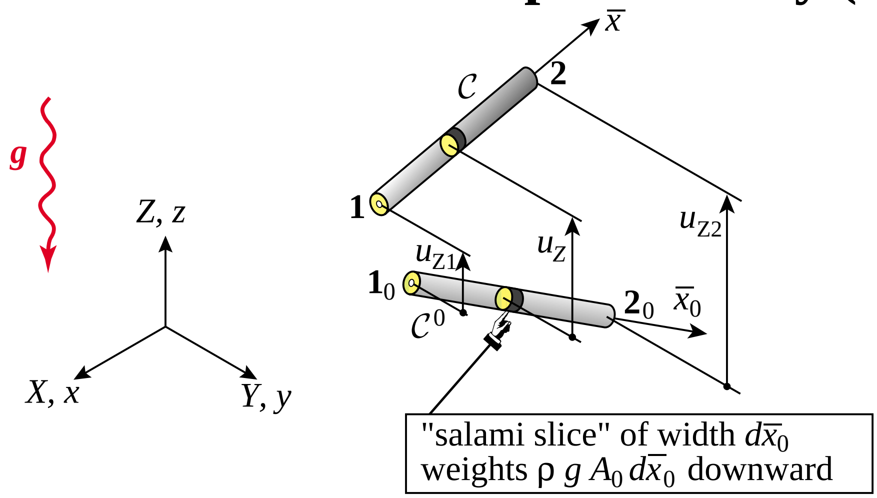


Consider a 2-node, TL bar element moving in 3D space. The bar is immersed in a gravity field of constant strength g acting along the global $-Z$ axis, as shown in the Figure. The bar has length L_0, cross section area A_0 and mass density ρ in the reference configuration. The local (element) coordinate systems are labeled

$\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ **in the reference configuration** $\mathcal{C}^0$

$\bar{x}$, $\bar{y}$, $\bar{z}$ **in the current configuration** $\mathcal{C}$

Potential Force Example: Gravity (2)



The distinction between local coordinate systems is introduced here as it becomes important later. Take a differential length element of bar $d\bar{x}_0$ in $\mathcal{C}^0$. This moves to a corresponding position in $\mathcal{C}$, with a vertical displacement of u_Z with respect to $\mathcal{C}^0$. See Figure. The differential of work associated with this motion is

$$dW = \text{weight-force . distance} = -\,\rho\, g\, A_0 u_Z\, d\bar{x}_0$$

(The – sign comes from weight-force and displacement being in opposite directions)

Potential Force Example: Gravity (3)

The external work potential of the bar element is obtained by linearly interpolating the vertical displacement u_z, which is a linear function of $\bar{x}_0$, and integrating dW over the length L_0. The result is

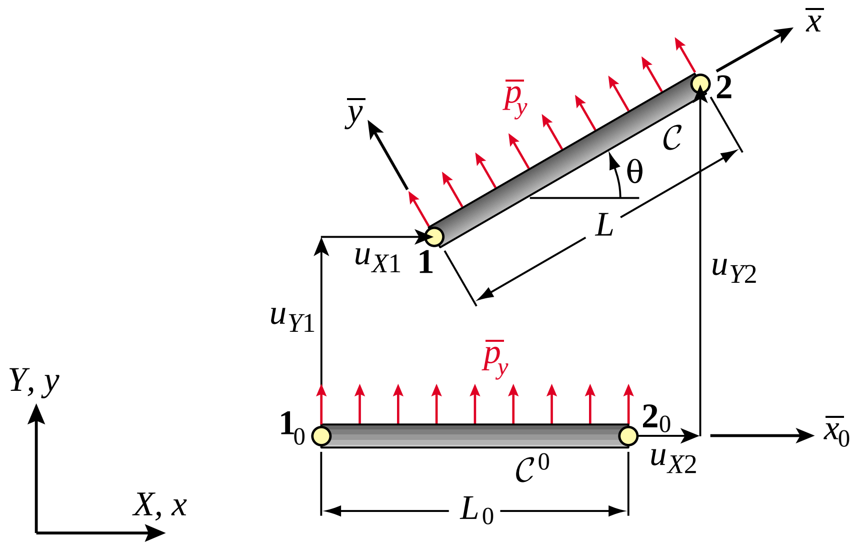
$$W = -\rho\, g\, A_0 L_0\, \tfrac{1}{2}\,(u_{z1} + u_{z2}) + C$$

in which C is an arbitrary constant. As usual in the TL description, all quantities are referred to the reference configuration. The external force vector can be obtained by differentiating wrt u:

$$\mathbf{f}_g = \frac{\partial W}{\partial \mathbf{u}} = \begin{bmatrix} \partial W/\partial u_{x1} \\ \partial W/\partial u_{y1} \\ \partial W/\partial u_{z1} \\ \partial W/\partial u_{x2} \\ \partial W/\partial u_{y2} \\ \partial W/\partial u_{z2} \end{bmatrix} = -\tfrac{1}{2}\rho A_0 L_0 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(Of course this can be also be readily obtained from statics)

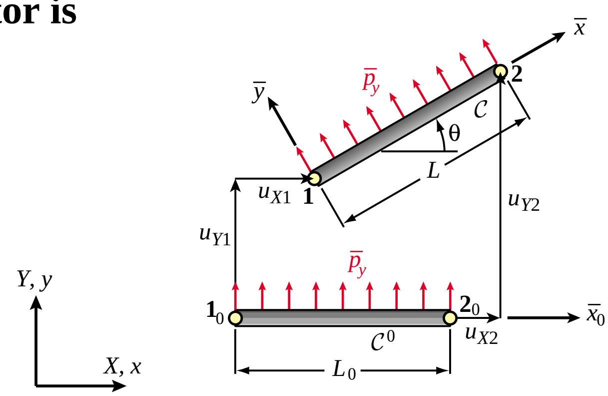
Follower Load on 2-Node Bar Moving in 2D



Follower Load on 2-Node Bar Moving in 2D (2)

From statics the external force vector is

$$\mathbf{f} = \tfrac{1}{2}\,\bar{p}_y\,L \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$



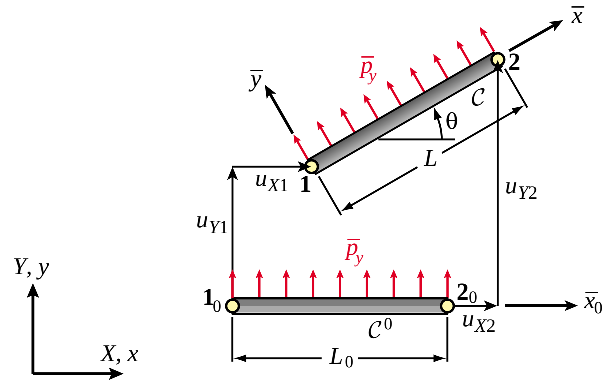
From geometry

$$\cos\theta = \frac{L_0 + u_{X21}}{L} \qquad \sin\theta = \frac{u_{Y21}}{L}$$

in which

$$u_{X21} = u_{X2} - u_{X1} \qquad u_{Y21} = u_{Y2} - u_{Y1}$$

Follower Load on 2-Node Bar Moving in 2D (3)



Expressing sine and cosine of tilt angle θ in terms of bar length and node displacements gives

$$\mathbf{f} = \tfrac{1}{2}\,\bar{p}_y \begin{bmatrix} -u_{Y21} \\ L_0 + u_{X21} \\ 0 \\ -u_{Y21} \\ L_0 + u_{X21} \\ 0 \end{bmatrix}$$

Follower Load on 2-Node Bar Moving in 2D (4)

Take now the (negated) partial of the force vector wrt to the state vector u:

$$\mathbf{K}_L = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \tfrac{1}{2}\,\bar{p}_y \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is called the load stiffness matrix. It arises from displacement dependent loads. For the follower load $\mathbf{K}_L$ is unsymmetric.
A consequence is that $\mathbf{f}$ cannot be derived from a potential.

Pressurized 3-Node Membrane Plate Triangle Moving in 3D

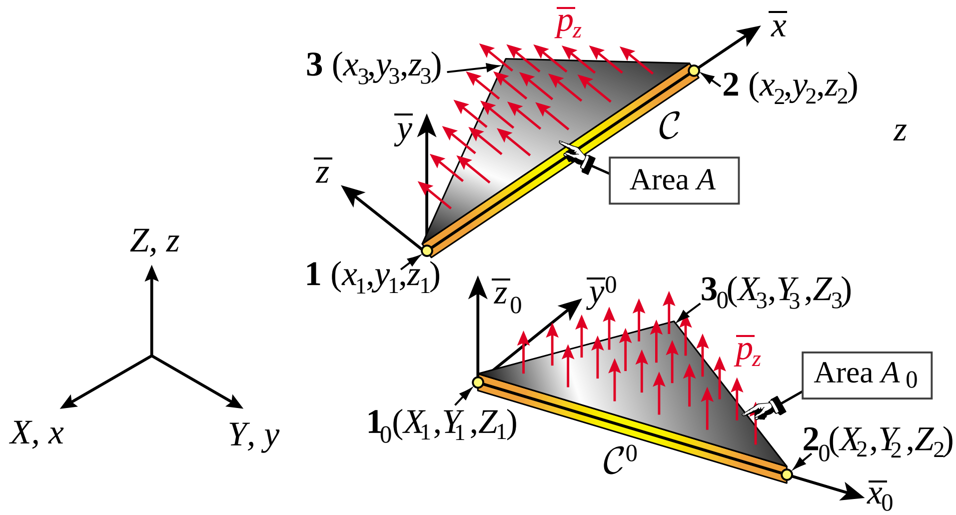
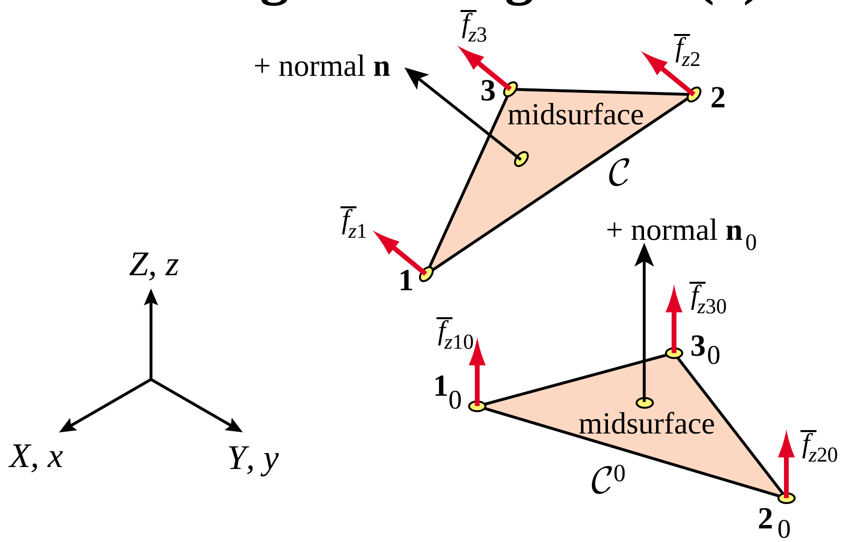


Plate is very thin: no bending resistance. Appropriate for problems such as inflating balloons and vehicle sails.

Nine DOF: three (X,Y,Z) displacements @ each corner.

Pressurized 3-Node Membrane Plate Triangle Moving in 3D (2)

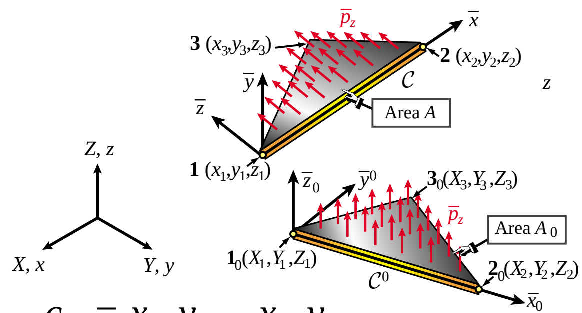


Procedure to get force vector:

(1) Total force on element is pressure x area, directed along normal to plate midsurface
(2) Lump total force into 3 equal forces, and assign to each corner node
(3) Project corner nodal forces on three global axes (X,Y,Z) using the direction cosines of the normal vector

Pressurized 3-Node Membrane Plate Triangle Moving in 3D (3)

To get normal direction in the current configuration $\mathcal{C}$, compute the direction numbers of the normal



$$a_n = y_{13}\, z_{21} - y_{12}\, z_{31}, \quad b_n = x_{21} z_{13} - x_{31} z_{12}, \quad c_n = x_{13} y_{21} - x_{12} y_{31}$$

These have dimensions of length squared. The triangle area in $\mathcal{C}$ is given by

$$A = \tfrac{1}{2} S_n \quad \textbf{in which} \quad S_n = +\sqrt{a_n^2 + b_n^2 + c_n^2}$$

The direction cosines of the normal are

$$\alpha_n = a_n/S_n, \quad \beta_n = b_n/S_n, \quad \gamma_n = c_n/S_n$$

Pressurized 3-Node Membrane Plate Triangle Moving in 3D (4)

Final result for force vector:

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{z1} \\ f_{x2} \\ f_{y2} \\ f_{z2} \\ f_{x3} \\ f_{y3} \\ f_{z3} \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\,A \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \alpha_n \\ \beta_n \\ \gamma_n \\ \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\,A \begin{bmatrix} a_n/S_n \\ b_n/S_n \\ c_n/S_n \\ a_n/S_n \\ b_n/S_n \\ c_n/S_n \\ a_n/S_n \\ b_n/S_n \\ c_n/S_n \end{bmatrix} = \tfrac{1}{3}\,\bar{p}_z\,A \begin{bmatrix} a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \\ a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \\ a_n/(2A) \\ b_n/(2A) \\ c_n/(2A) \end{bmatrix} = \tfrac{1}{6}\,\bar{p}_z \begin{bmatrix} a_n \\ b_n \\ c_n \\ a_n \\ b_n \\ c_n \\ a_n \\ b_n \\ c_n \end{bmatrix}$$

The derivation of the load stiffness $\mathbf{K}_L$ from this vector is left as an Exercise

Load Stiffness For Conservative System

Suppose that we have a conservative system with staging control parameter λ and state variables collected in vector $\mathbf{u}$. The external force potential depends on both λ and $\mathbf{u}$:

$$\Pi = U(\mathbf{u}) - W(\mathbf{u},\lambda)$$

Diffrentiating repeatedly with respect to the state:

$$\mathbf{r} = \frac{\partial \Pi}{\partial \mathbf{u}} = \frac{\partial U}{\partial \mathbf{u}} - \frac{\partial W}{\partial \mathbf{u}} = \mathbf{p} - \mathbf{f}$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

Load Stiffness For Conservative System (2)

The partial ∂p/∂u gives $K_M + K_G$, the material plus geometric stiffness, as discussed in previous Chapters. The last term gives the conservative load stiffness:

$$\mathbf{K}_L = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = -\frac{\partial^2 W}{\partial \mathbf{u}^2}$$

This matrix is obviouly symmetric because it is the (negated) Hessian of W(u,λ) wrt u. Consequently

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_L$$

is still symmetric. The static stability tests, in particular vanishing of the determinant of K, still apply.

Load Stiffness For Nonconservative System

Consider next a system subjected to both conservative and nonconservative loads. The total residual can be written

$$\boxed{\mathbf{r} = \mathbf{p} - \mathbf{f}_c - \mathbf{f}_n}$$

Here $\mathbf{f}_c$ are the conservative forces, which can be derived from a external work potential W, whereas $\mathbf{f}_n$ collects the nonconservative forces, which cannot. Differentiation wrt to the state gives the tangent stiffness matrix as

$$\boxed{\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}} = \mathbf{K}_M + \mathbf{K}_G + \mathbf{K}_{Lc} + \mathbf{K}_{Ln}}$$

Here the load stiffness $\mathbf{K}_L$ is split into a conservative part $\mathbf{K}_{Lc}$, which is symmetric, and a nonconservative part $\mathbf{K}_{Ln}$, which is not.

Load Stiffness For Nonconservative System (2)

In practice one does not always know *a priori* whether the loading is conservative or not. For example, hydrostatic pressure loads on a submerged structure such as a offshore platform or a submarine, are actually conservative, even for finite changes in geometry.

To find out proceed as follows:

1. **Derive the force vector f for e.g., one element, using statics**
2. **Take partials wrt state DOF to obtain the load stiffness**
3. **Check for symmetry**

If unsymmetric, try splitting the load stiffness into symmetric and nonsymmetric (often this is not unique), and work your way back to a potential for the symmetric part.