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Nonlinear Bifurcation Analysis

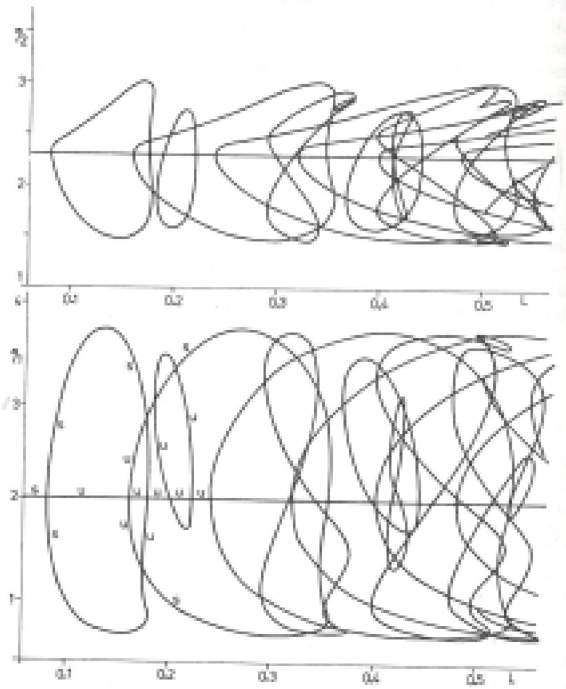


Figure 3.2 Dependence of the steady-state solutions on the parameter L; Brusselator model, Example 6, Chapter 1, $A = 2$, $B = 4.6$, BC 2, $D_x = 0.0016$, $D_y = 0.008$, $\eta_1 = x(0)$, $\eta_2 = y(0)$. s—stable, u—unstable.

Bifurcation Analysis Levels

1. Locate

Find where *B* occurs while tracing a response

2. Determine subspace where "the action is"

Having located *B*, find y (particular solution) and z (homogeneous solution = null eigenvector in case of a simple bifurcation)

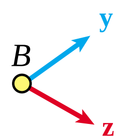
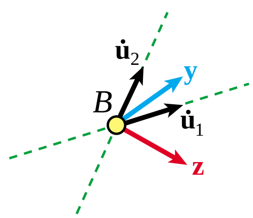
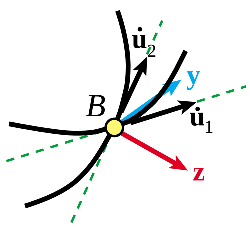
3. Branching analysis

Having located *B* and determined y and z, find the directions $\dot{\mathbf{u}}_1$ and $\dot{\mathbf{u}}_2$ of tangents to the equilibrium paths (branches) that pass through *B*. Requires analysis of $\ddot{\mathbf{r}} = 0$

4. Branch curvature analysis

Having located *B*, and determined y, z, $\dot{\mathbf{u}}_1$ and $\dot{\mathbf{u}}_2$ find the curvatures of the equilibrium paths (branches) that pass through *B*. Requires analysis of $\dddot{\mathbf{r}} = 0$

Bifurcation Analysis Levels: Pictures

Level	Diagram	Analysis Type	Rate Equation Order
1		**Locate**	*1 (factorization)*
2		**Determine active subspace**	*1 (eigensolution)*
3		**Determine branch tangents at** B	*2 (if regular)*
4		**Determine branch curvatures at** B	*3 (if regular)*

Can Commercial FEM Codes Do These Levels?

Level 1: yes
Level 2: yes
Level 3: very difficult
Level 4: impossible

Levels 3-4 can be directly done by hand, or with CAS help, only for very simple problems.

For more complicated ones, there are indirect ways to try such as injecting numeric perturbations.
Nonetheless, it helps to know the theory to guide the computations (there is no such thing as a "black box" for this kind of analysis)

Governing Equations Recap

Total residual equations:

$$\mathbf{r}(\mathbf{u}, \lambda) = \mathbf{0}$$

Parametric representation of the solutions (with t **as pseudotime)**

$$\mathbf{u} = \mathbf{u}(t) \qquad \lambda = \lambda(t)$$

Two important special choices for t **:**

$$t = \lambda \qquad t = s$$

which lead to λ-parametrized and arclength forms, respectively

Rate Forms

Rate forms of the governing residual equations are obtained by repeated differentiation of the total residual wrt pseudotime t **:**

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{K}\dot{\mathbf{u}} - \mathbf{q}\dot{\lambda} = \mathbf{0}\\ \ddot{\mathbf{r}} &= \mathbf{K}\ddot{\mathbf{u}} + \dot{\mathbf{K}}\dot{\mathbf{u}} - \dot{\mathbf{q}}\dot{\lambda} - \mathbf{q}\ddot{\lambda} = \mathbf{0}\\ \dddot{\mathbf{r}} &= \mathbf{K}\dddot{\mathbf{u}} + \dot{\mathbf{K}}\ddot{\mathbf{u}} + \ddot{\mathbf{K}}\dot{\mathbf{u}} - \mathbf{q}\dddot{\lambda} - \dot{\mathbf{q}}\ddot{\lambda} - \ddot{\mathbf{q}}\dot{\lambda} = \mathbf{0}\end{aligned}$$

in which

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \mathbf{u}}, \qquad \mathbf{q} = -\frac{\partial \mathbf{r}}{\partial \lambda} = -\mathbf{r}'$$

$\dot{\mathbf{r}} = \mathbf{0}$ **is a system of first-order rate equations, also called the incremental stiffness equations or simply stiffness equations.** $\ddot{\mathbf{r}} = \mathbf{0}$ **is a system of second-order rate equations, also called the stiffness rate equations. And so on. For the branching analysis undertaken here we will go up to** $\ddot{\mathbf{r}} = \mathbf{0}$**.**

Stiffness And Incremental Load Rates

In the second-order rate equation

$$\ddot{\mathbf{r}} = \mathbf{K}\ddot{\mathbf{u}} + \dot{\mathbf{K}}\dot{\mathbf{u}} - \dot{\mathbf{q}}\dot{\lambda} - \mathbf{q}\ddot{\lambda} = \mathbf{0}$$

the stiffness rate and incremental load arrays can be compactly expressed as

$$\dot{\mathbf{K}} = \mathbf{L}\dot{\mathbf{u}} + \mathbf{N}\dot{\lambda} \qquad \dot{\mathbf{q}} = -(\mathbf{N}\dot{\mathbf{u}} + \mathbf{a}\dot{\lambda})$$

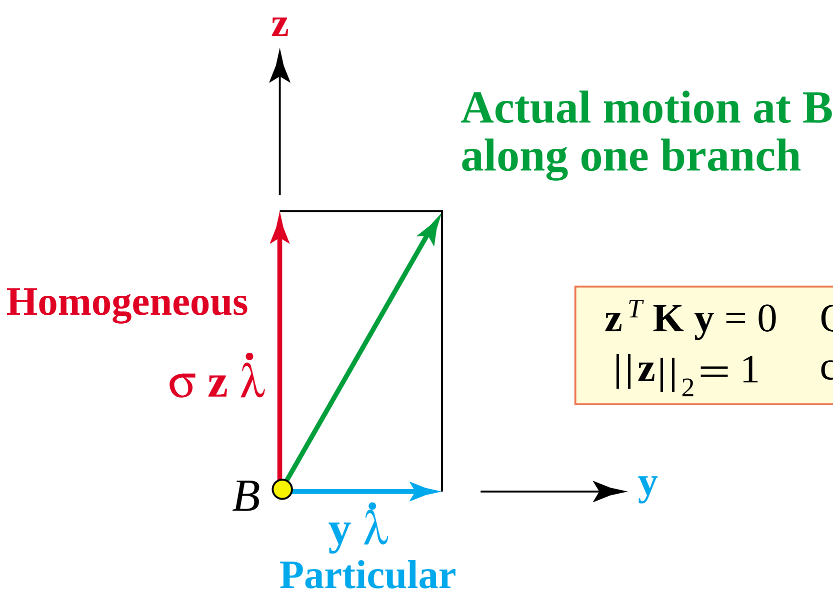
The entries of these matrices and vectors can be written in indicial form as

$$L_{ijk} = \frac{\partial^2 r_i}{\partial u_j \partial u_k} \qquad N_{ij} = \frac{\partial^2 r_i}{\partial u_j \partial \lambda} = \frac{\partial K_{ij}}{\partial \lambda} \qquad a_i = \frac{\partial^2 r_i}{\partial \lambda \partial \lambda} = \frac{\partial q_i}{\partial \lambda}$$

Note that L is a three-dimensional array that may be called a "cubic" matrix. Postmultiplying a cubic matrix by a vector yields an ordinary matrix. For example $\mathbf{L}\,\dot{\mathbf{u}}$ in the expression of $\dot{\mathbf{K}}$ is a matrix

State Decomposition At Bifurcation

Recall from Chapter 28 the state decomposition at *B*



$\mathbf{z}^T \mathbf{K}\, \mathbf{y} = 0$	Orthonormality
$\|\|\mathbf{z}\|\|_2 = 1$	conditions

Finding the Null Eigenvector (Buckling Mode Shape)

To find z, carry out an eigenvalue analysis of the tangent stiffness matrix K at bifurcation point B. If this is an simple bifurcation point (which we will asume), z is the null eigenvector of K evaluated so that

$$\mathbf{K}\,\mathbf{z} = \mathbf{0}$$

This eigenvector is guaranteed to have real entries since K is symmetric. Normalize this vector to unit length so that

$$\|\mathbf{z}\|_2 = 1$$

The amplitude of this eigenvector in the state rates near *B* **is called** σ.

Physically this eigenvector is often called the buckling mode shape but the name is not always accurate as regards physics

Control-State Rate Relation At Bifurcation

To carry out Level 2 analysis, recall from Chapter 28 that the linearized relation between control and state pseudotime rates measured from *B* can be expressed as a linear combination of y and z,

$$\dot{\mathbf{u}} = (\,\mathbf{y} + \sigma\,\mathbf{z})\,\dot{\lambda} \qquad \mathbf{z}^T\mathbf{K}\,\mathbf{y} = \mathbf{y}^T\mathbf{K}\,\mathbf{z} = 0$$

Here z is (as discussed in the previous slide) the unit-length-normalized null eigenvector at *B* (often called the buckling mode shape), whereas y is the particular solution satisfying

$$\mathbf{K}\,\mathbf{y} = \mathbf{q}$$

as well as the orthonormality condition shown above; all quantities being evaluated at *B*

First-Order Rate Equations Do Not Cut It for Level 3

Can we do Level 3 branching analysis with $\dot{r} = 0$? This is the subject of HW Exercise 31.3. The analysis gives

$$\boxed{\sigma = 0/0}$$

Consequently the null eigenvector (buckling mode) amplitude is indeterminate, and the direction of the branches is undefined.

To get definite branching information it is necessary to proceed to the second-order rate equations. Good news: this will usually work (as long as the quadratic equation derived later does not fully degenerate). Bad news: we get more than one solution.

Level 3 Branching Analysis

To do Level 3 branching analysis we start from

$$\ddot{\mathbf{r}} = \mathbf{K}\ddot{\mathbf{u}} + \dot{\mathbf{K}}\dot{\mathbf{u}} - \dot{\mathbf{q}}\dot{\lambda} - \mathbf{q}\ddot{\lambda} = \mathbf{0}$$

Premultiply both sides by $\mathbf{z}^T$. Taking account of the bifurcation conditions

$$\mathbf{K}\,\mathbf{z} = \mathbf{z}^T\mathbf{K} = 0 \quad \mathbf{q}^T\mathbf{z} = \mathbf{z}^T\mathbf{q} = 0$$

we get

$$\mathbf{z}^T\dot{\mathbf{K}}\dot{\mathbf{u}} - \mathbf{z}^T\dot{\mathbf{q}}\dot{\lambda} = 0$$

Replacing the stiffness and incremental load rates by their compact expressions yields

$$\mathbf{z}^T(\mathbf{L}\dot{\mathbf{u}} + \mathbf{N}\dot{\lambda})\dot{\mathbf{u}} + \mathbf{z}^T(\mathbf{N}\dot{\mathbf{u}} + \mathbf{a}\dot{\lambda})\dot{\lambda} = 0$$

Level 3 Branching Analysis (cont'd)

Substituting the state rates by the state decomposition gives

$$\mathbf{z}^T\left[\mathbf{L}(\mathbf{y}+\sigma\mathbf{z})\dot{\lambda}+\mathbf{N}\dot{\lambda}\right](\mathbf{y}+\sigma\mathbf{z})\dot{\lambda}+\mathbf{z}^T\left[\mathbf{N}(\mathbf{y}+\sigma\mathbf{z})\dot{\lambda}+\mathbf{a}\dot{\lambda}\right]\dot{\lambda}=0$$

Collecting terms in the control rate and requesting a nontrivial solution gives the quadratic equation in σ

$$a\sigma^2+2b\sigma+c=0$$

in which

$$\begin{aligned} a &= \mathbf{z}^T\mathbf{L}\mathbf{z}\mathbf{z} \\ b &= \mathbf{z}^T\left[\mathbf{L}\mathbf{z}\mathbf{y}+\mathbf{L}\mathbf{y}\mathbf{z}+2\mathbf{N}\mathbf{z}\right] \\ c &= \mathbf{z}^T\left[\mathbf{L}\mathbf{y}\mathbf{y}+2\mathbf{N}\mathbf{y}+\mathbf{a}\right] \end{aligned}$$

Level 3 Branching Analysis (cont'd)

This quadratic equation generally provides two roots: σ_1 and σ_2. In what follows we shall assume that the roots are real (see Chapter for discussion) . Substitution into

$$\dot{\mathbf{u}} = (\,\mathbf{y} + \sigma\,\mathbf{z})\,\dot{\lambda}$$

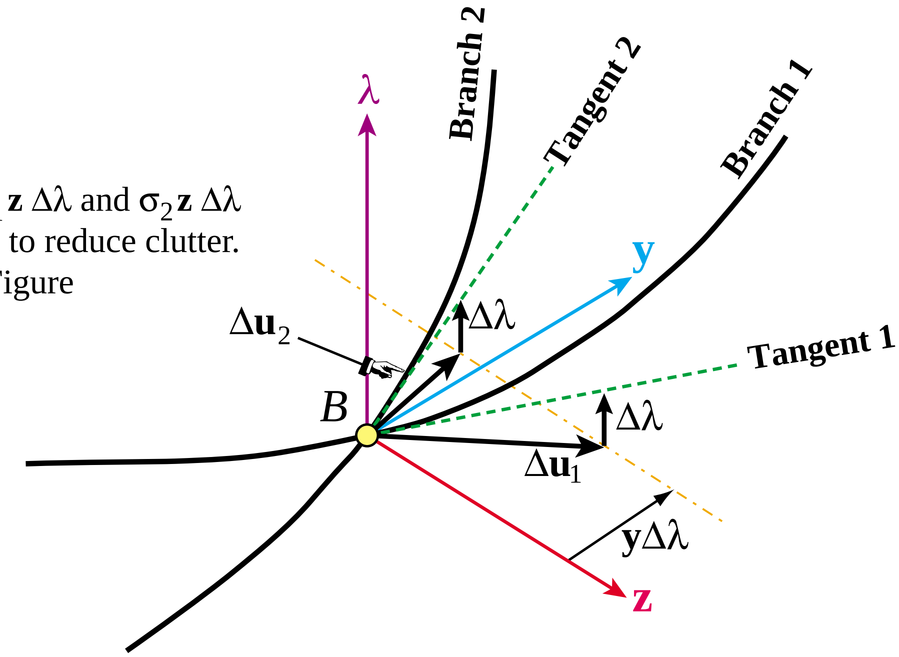
furnishes the two branch directions at the bifurcation point:

$$\dot{\mathbf{u}} = (\,\mathbf{y} + \sigma_1\mathbf{z})\,\dot{\lambda} \qquad \dot{\mathbf{u}} = (\,\mathbf{y} + \sigma_2\mathbf{z})\,\dot{\lambda}$$

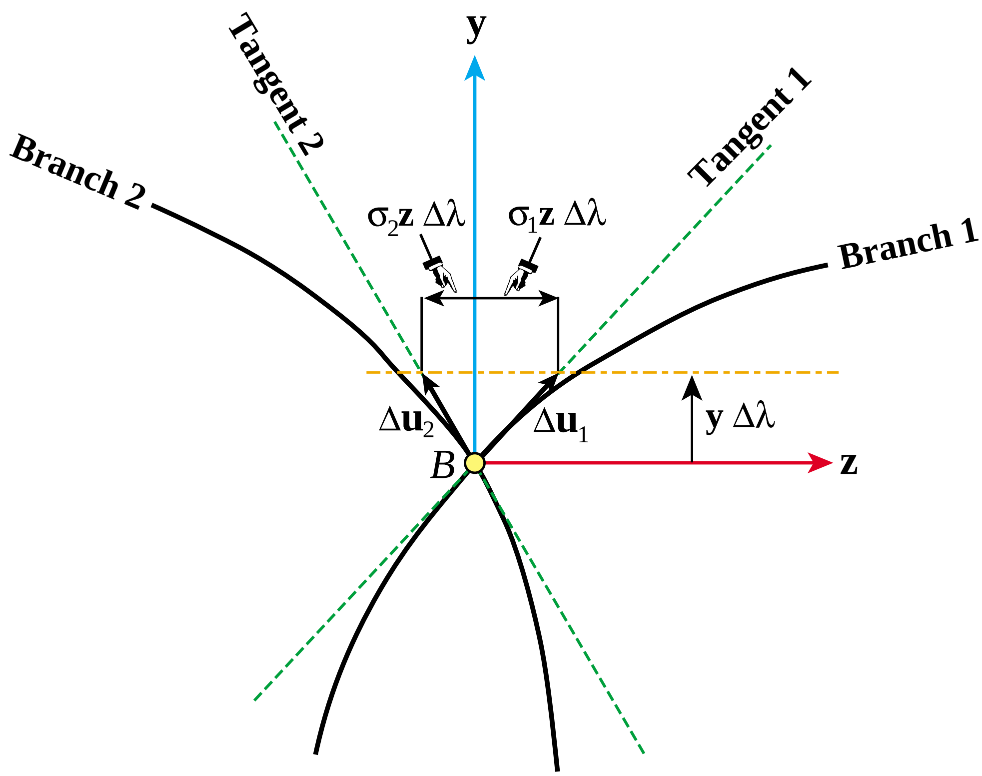
These are sketched in the next two slides, first in the three dimensional space (y, z, λ), then projected on the (y, z) plane.

Branching Directions In (y, z, λ) Space

Vectors $\boldsymbol{\sigma}_1 \mathbf{z}\, \Delta\lambda$ and $\boldsymbol{\sigma}_2 \mathbf{z}\, \Delta\lambda$
not shown to reduce clutter.
See next Figure



Branching Directions On (y, z) Plane



Special Forms Of The σ–Quadratic Equation

(questions to class - See Chapter for discussion)

$$\boxed{a\sigma^2 + 2b\sigma + c = 0}$$

What happens if $a = 0$ **?**

What happens if $a = c = 0$ **?**

Is $a = b = 0$ **possible?**

What happens if $a = b = c = 0$ **?**

What Happens At A Multiple (a.k.a. Compound) Bifurcation Point?

Suppose now that two bifurcation points coincide. There are now two independent null eigenvectors:

$$\boxed{\mathbf{z}_1 \quad \mathbf{z}_2}$$

which together span a null eigenspace Z **of dimension two (in linear algebra terminology, the kernel space of K evaluated at bifurcation point** B **has dimension 2)**

Assuming all roots of the branching equations are real, how many equilibrium branches go through B**?**

Unexpected Answer

If the bifurcation point has multiplicity M, the number of branches crossing there ranges from

$$2M-1 \quad \text{to} \quad (3^M-1)/2$$

depending on local symmetries. Results were published by Sewell (1968), Johns and Chilver (1971) for general Jacobians (not necessarily derivable from potentials)

The complexity of the analysis grows enormously as M increases:

For M=2 the range is 3 to 4
For M=5 (NASA benchmark: composite panel with a rectangular cutout) the range is 31 to 121