32

Qualitative Analysis of Critical Points

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§32.1. General Notion of Stability

In Chapter 28 stability was informally defined as the ability of a physical system to return to equilibrium when disturbed. If the equilibrium is static in nature, we speak of *static stability*. For a more precise definition concerning a mechanical system, let's hear Dirichlet:[1]

"The equilibrium [of a mechanical system] is stable if, in displacing the points of the system from their equilibrium positions by an infinitesimal amount and giving each one a small initial velocity, the displacements of different points of the system remain, throughout the course of the motion, contained within small prescribed limits"

Some essential ingredients of this definition are:

(1) Stability is a quality of *one* solution — an equilibrium configuration of the system. (We cannot generally state *a priori* that a system, *as a whole*, is stable or unstable.)

(2) The problem of ascertaining the stability of an equilibrium solution concerns the "neighborhood" of that solution and is therefore a *local one*.

(3) The concept of stability is inherently *dynamic* in nature. But for a *conservative system* time can be "factored out" of the problem, and we are left with *static* criteria. These are much simpler to apply.

§32.2. Stability of a Discrete Conservative System

As discussed in Chapter 27, the static stability of a *conservative* mechanical system can be tested completely using a *static* criterion. Such criterion, often referred to as the *Euler stability test*, the *energy test*, or the *method of adjacent states*, relates to the positive definiteness character of the second variation of the potential energy.

We know that a stationary value of the total potential energy with respect to the state variables is necessary and sufficient for the *equilibrium* of the system. Proceeding one step further, a complete *relative minimum* of the total potential energy is necessary and sufficient for the *stability* of an equilibrium state.

For a *discrete* system with a finite number of degrees of freedom the criterion can be enunciated in terms of the definiteness of the *tangent stiffness matrix* **K** if all state variables are of displacement type, which we assume in the sequel (see Remark below). For a conservative system we know that **K** is a *symmetric* matrix.

[1] As it appears in his Appendix to the German translation of Lagrange's *Mécanique Analytique* [453].

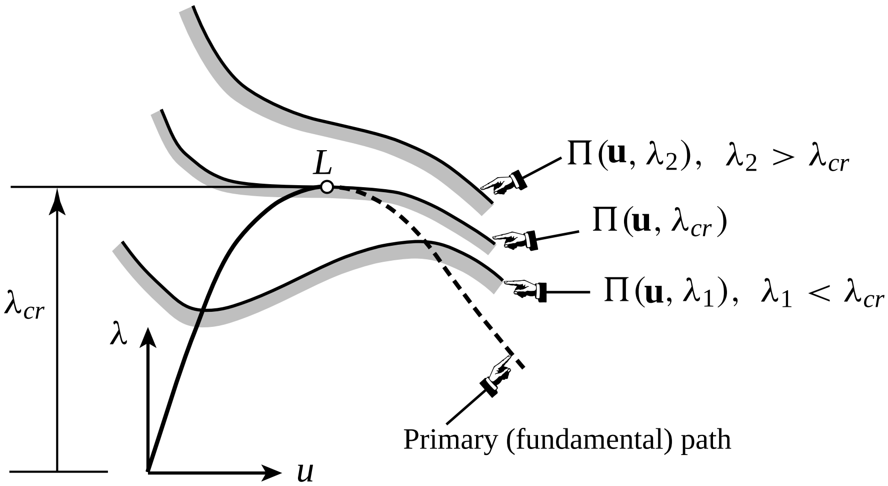

FIGURE 32.1. Transformation of potential energy at a limit point.

Remark 32.1. The restriction to systems with displacement state variables aims to exclude those in which Lagrange multipliers are carried along as degrees of freedom. For such systems the criterion applies upon eliminating the multipliers, but such elimination is often messy and would complicate the exposition.

The stability criterion for a conservative system is summarized in the following table.

If **K** *evaluated at an equilibrium position is*	*The potential energy* Π *at that position has a*	*Then the equilibrium position is*
positive definite	strict minimum	**stable**
positive semidefinite	cylindrical or inflexion point	**neutrally stable**
indefinite	saddle point	**unstable**

If the eigenvalues of **K** are easily available a test for stability is immediate.[2] If all eigenvalues are greater than zero, the matrix is positive definite and the equilibrium is stable. If one or more eigenvalues are zero and the rest positive, the equilibrium is neutrally stable. If one or more eigenvalues are negative, the equilibrium is unstable.

In practice an eigenvalue test can be recommended only for small matrices, say of order less than 20 or so. For larger matrices the same information can be obtained more economically by decomposing **K** using triangular factorization or Gauss elimination, as discussed in Remark **?**. If all pivots are positive, the equilibrium is stable. If at least one pivot is negative, the equilibrium is unstable. The border case of neutral stability is more difficult to detect in the presence of rounding errors.

[2] See **?** for computational details.

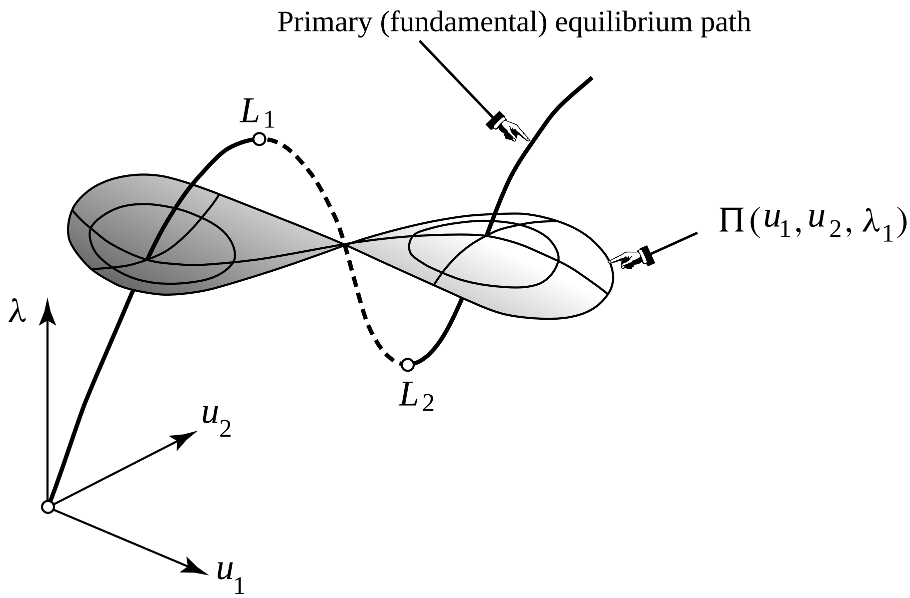

FIGURE 32.2. Typical potential energy surface in snap-through response.

Is it necessary to apply these criteria to each equilibrium configuration of the system? No. Since stability can change to instability (or vice-versa) only at critical points, it is generally sufficient to test only one solution between critical points to ascertain whether the path is stable or unstable.[3]

§32.3. Stability Transformation at a Limit Point

As discussed in previous sections, there is a close relationship between equilibrium configurations, occurrence of critical points, and the stability of the system. We now examine qualitatively, following the monographs of Thompson and Hunt [769,771], four types of critical points from the standpoint of the variation of the *total potential energy* in the neighborhood of equilibrium states. This is done with the typical response plots in which the control parameter λ is the vertical axis while a representative displacement u or deformation mode amplitude is shown along the horizontal axis as state parameter.

Drawing conventions are as follows: heavy lines represent equilibrium paths, continuous lines denoting stable paths while broken lines denote unstable paths. Plots of total potential energy $\Pi(u, \lambda)$ at various *fixed* values of λ are shown as "shaded profile" energy surfaces. These surfaces deform as the parameter λ changes. Equilibrium configurations correspond to stationary points of Π with respect to u. Strong minima (maxima) of this surface are associated with stable (unstable) equilibrium configurations.

We first consider the case of a *limit point*, which is shown in Figure 32.1. The fundamental equilibrium path that starts from the origin (the reference configuration $\mathbf{u} = 0$, $\lambda = 0$) is initially stable. Stability is lost when the local maximum at $\lambda = \lambda_{cr}$ is reached. $\lambda = \lambda_{cr}$. A "snap-through" response of this form is characteristic of shallow arches and domes. At a *fixed* value $\lambda = \lambda_1$ less than λ_{cr} the total potential energy $\Pi(u, \lambda_1)$ has a minimum with respect to the state parameter

[3] Exceptions to this may occur in pathological cases (for example, a whole path is neutrally stable) that require further investigation.

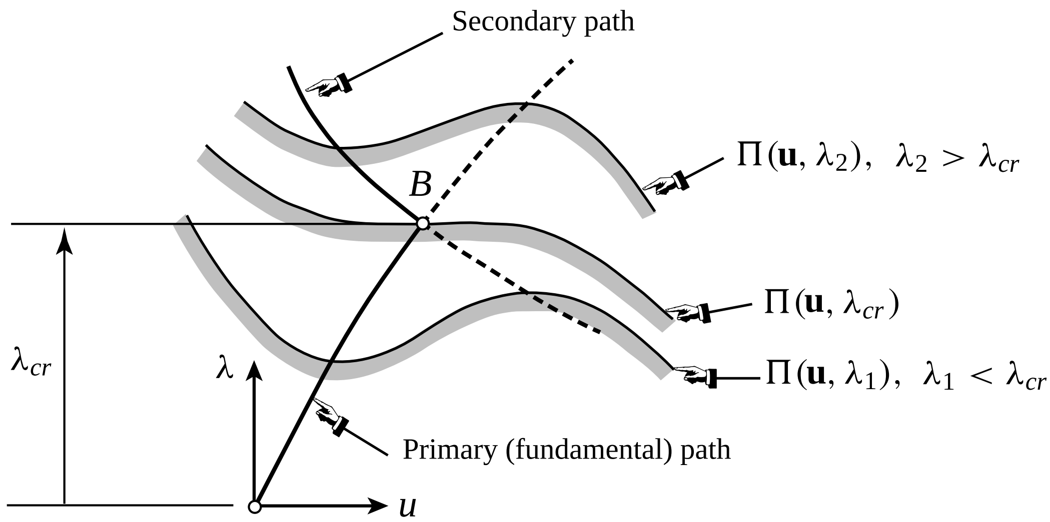

FIGURE 32.3. Asymmetric bifurcation point.

u on the stable rising region of the path and a maximum on the unstable falling region. As the prescribed value of λ is increased the maximum and minimum approach each other and coalesce when $\lambda = \lambda_{cr}$. At this critical point the total potential energy $\Pi(u, \lambda)$ has an horizontal point of inflexion. At a higher value of λ, say $\lambda_2 > \lambda_{cr}$, there are no local equilibrium states and the total potential energy $\Pi(u, \lambda_2)$ has no stationary point. The critical equilibrium state is seen to be itself unstable, and the absence of local equilibrium states at values of λ greater than λ_{cr} implies that a physical system under slowly increasing λ will eventually snap-through dynamically. Limit points are generally insensitive to imperfections.

A more general schematic diagram is shown in three dimensions in Figure 32.2 on a plot of λ against *two* state parameters, v_1 and v_2. This plot includes a remote rising region of the equilibrium path since this is often encountered with this type of behavior. A total potential energy surface $\Pi(v_1, v_2, \lambda)$ is drawn for a *fixed* value of $\lambda < \lambda_{cr}$. As λ is slowly increased through its critical value the system will "snap through" dynamically, eventually stop, and initiate a large amplitude, nonlinear vibration about the remote stable equilibrium path. In the presence of some damping the system will eventually rest on that path.

These figures illustrate the physics well but if we are dealing with a system with many degrees of freedom care must be taken in drawing conclusions from these *schematic* figures. On an actual plot of λ against one of the v_i's, the limit point is *normally* seen as a smooth maximum, but it must be realized that for a *certain* choice of the state parameter v_i the point *might* appear as a sharp cusp. The smooth maximum of a path in three-dimensional space can for example be seen as a cusp if the eye is directed along the horizontal tangent to the path.

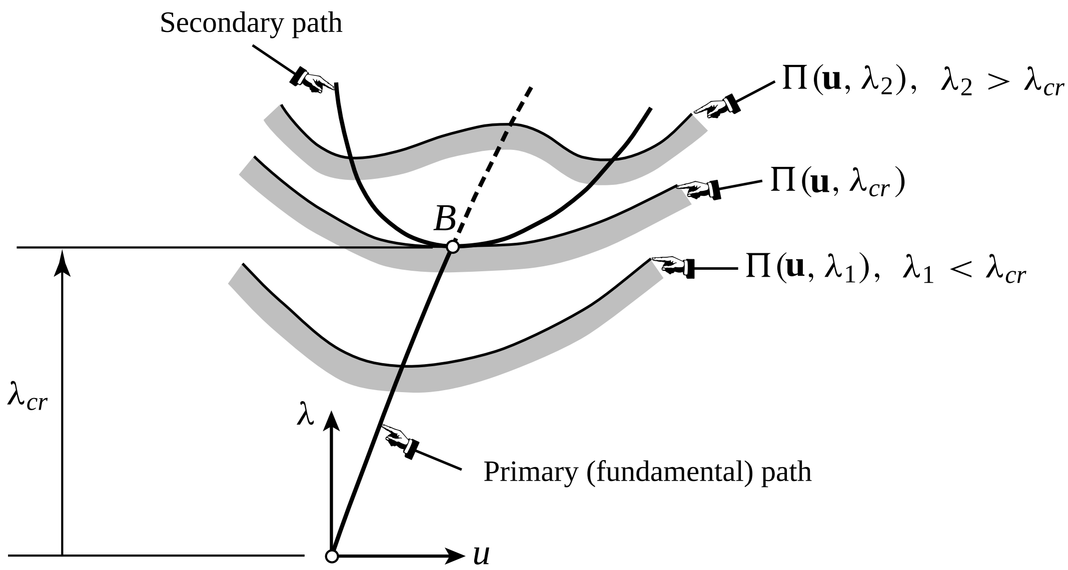

FIGURE 32.4. Stable-symmetric bifurcation point.

§32.4. Stability Exchange at Bifurcation Points

After the limit point we consider bifurcation or branching points. We cover the three most common types of bifurcation: asymmetric, stable-symmetric, and unstable symmetric.

§32.4.1. Asymmetric Bifurcation

Figure 32.3 illustrates the case of an *asymmetric point of bifurcation*. The initially stable fundamental equilibrium path that emanates from the origin loses its stability on intersecting a distinct and continuous secondary (post-buckling) equilibrium path. The intersection point B is a critical point of bifurcation type. An asymmetric bifurcation point is characterized by the fact that both paths have a *nonzero* slope with respect to λ at B.

With varying λ the paths exhibit a phenomenon called *exchange of stability*. For $\lambda_1 < \lambda_{cr}$ the total potential energy $\Pi(u, \lambda_1)$ has a minimum with respect to u on the stable region of the fundamental path and a maximum with respect to u on the unstable region of the post-buckling path. As λ is increased the maximum and minimum finally coalesce so that at $\lambda = \lambda_{cr}$ the total potential energy $\Pi(u, \lambda_{cr})$ has a horizontal point of inflexion at the critical equilibrium state. At λ values over the critical one the maximum and minimum exchange places. Since an unstable branch emanates from B, the critical equilibrium state is *unstable*. In the presence of small disturbances a physical system under slowly increasing λ would snap dynamically from this critical equilibrium state despite the existence of stable equilibrium states at higher values of λ.

Critical points of this type are moderately to highly sensitive to the presence of loading or fabrication imperfections.

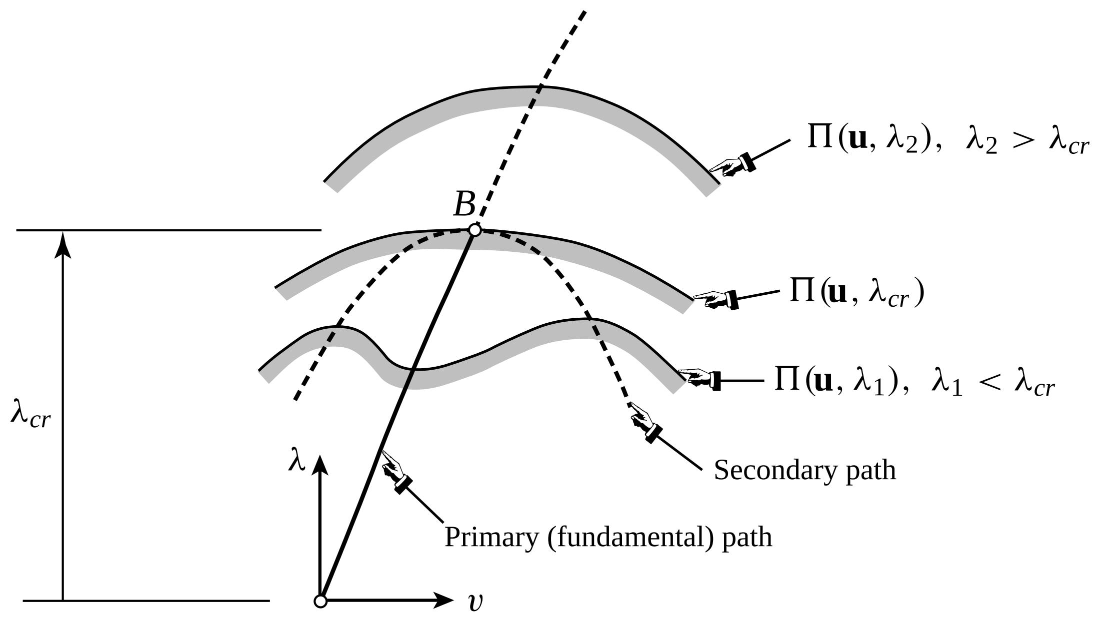

FIGURE 32.5. Unstable-symmetric bifurcation point.

§32.4.2. Stable-Symmetric Bifurcation

Symmetric bifurcation points are characterized by the fact that the intersecting path has *zero* slope with respect to the control parameter at B. These points may be categorized into stable and unstable, depending on whether the intersecting post-buckling) path is "rising" or "falling".

Figure 32.4 depicts the case of an *stable-symmetric* point of bifurcation. Here a fundamental equilibrium path rising monotonically from the reference state is seen to intersect a stable rising secondary (post-buckling) path that passes smoothly through the critical equilibrium state with zero slope. The continuation of the fundamental path beyond B is unstable. The total potential energy $\Pi(u, \lambda_1)$, where $\lambda_1 < \lambda_{cr}$, has a single stationary value with respect to u, namely the minimum on the stable region of the fundamental path, and as the value of λ is increased this minimum is transformed into two minima and one maximum. The critical equilibrium state is neutrally stable and the secondary path is stable, so a physical system under slowly increasing λ would exhibit no dynamic snap but would follow the stable rising post-buckling path, the direction taken depending on the small disturbances or imperfections which are inevitably present.

Critical points of this type are insensitive to the presence of imperfections.

§32.4.3. Unstable-Symmetric bifurcation

The last configuration examined here is the *unstable-symmetric* point of bifurcation, shown in Figure 32.5. Here the fundamental path intersects an unstable falling path which as in the previous case has a zero slope at the critical equilibrium state. At a prescribed value of $\lambda = \lambda_1 < \lambda_{cr}$ the total potential energy $\Pi(u, \lambda_1)$ has now three stationary values with respect to u, namely two maxima on the unstable post-buckling or secondary path, and a minimum on the stable region of the fundamental path. As the figure shows, these three stationary points transform into a single maximum with increasing λ. The critical equilibrium state is seen to be unstable, so a physical system would snap dynamically from the critical equilibrium state, the direction taken depending on the postulated small disturbances or imperfections.

Critical points of this type are highly sensitive to the presence of structural or loading imperfections. Sometimes the sensitivity is extreme, as in the classical case of the axially compressed cylindrical shell discussed in §31.6.2.