

29

Linearized Prebuckling: Formulation

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§29.1. Introduction

This Chapter covers the simplest practical method for analyzing structural stability: Linearized Prebuckling, acronymed as LPB. Achieving that simplicity requires making several behavioral assumptions. These are discussed in detail in the next Chapter. If those assumptions are satisfied, or at least acceptable, LPB leads to a *linear eigenvalue problem* that involve constant matrices. This eigenproblem can be processed by well tested linear algebra packages. Its solution provides critical loads as eigenvalues and buckling mode shapes as eigenvectors.

The key advantage is that the need for carrying out a full nonlinear analysis for finding critical points is completely bypassed. As previously discussed, that kind of analysis can be time consuming in terms of human effort as well as computationally demanding. Furthermore, it typically requires detailed information about the structure, which may not be available during preliminary design stages.

The gain in simplicity and speed is balanced by severe limitations. LPB can only provide information on bifurcation points located in an undeflected primary path, since deformations prior to buckling are neglected. Two consequences: LPB cannot detect limit points, and all information on postbuckling behavior is lost.

On the plus side: LPB buckling predictions are often sufficient for preliminary design purposes of many structures, particularly those in Civil Engineering. Ordinary safety factors may be sufficient to cover shortcomings. And more realistic verification analyses could be carried out if necessary after a design is firmly established.

To set procedural details we must distinguish the kind of stability analysis to be performed.

- *Equilibrium Analysis.* This was illustrated in the previous Chapter for lumped parameter models (rigid struts propped by springs). The LPB procedure is straightforward: either linearize the exact stability equations as in the examples of §28.5, or (equivalently) linearize the FBD as in the examples of §28.6.
- *FEM/DSM Analysis.* For FEM models LPB emerges naturally from the linearization of the *singular stiffness test* (SST). at the reference state. The eigenproblem is constructed using the material and geometric stiffness matrices evaluated there. The SST is described in §29.3.

§29.2. Stability Assessment Criteria

For *elastic*, geometrically-nonlinear structures under *static* loading we can distinguish the following criteria for stability assessment.

$$\text{Loading} \left\{ \begin{array}{l} \text{Conservative} \left\{ \begin{array}{l} \text{Static criterion: singular stiffness} \\ \text{Dynamic criterion: zero frequency} \end{array} \right. \\ \text{Nonconservative} \left\{ \begin{array}{l} \text{Dynamic criterion} \left\{ \begin{array}{l} \text{zero frequency (divergence)} \\ \text{frequency coalescence (flutter)} \end{array} \right. \end{array} \right. \end{array} \right. \quad (29.1)$$

This is similar to the “stability loss scenarios” list of Figure 28.2. But several changes may be noted. An important one is that both bifurcation and limit points can be handled with the *singular stiffness test* (SST) criterion, introduced in §5.2. The test is interpreted in §29.3 from a FEM standpoint as the source of the LPB method.

§29.3. Static Criterion

The *static criterion* is also known as *Euler's method*, since Euler introduced it in his famous investigations of the elastica published in 1744 [205]. Other names for it are *energy method*, *method of adjacent states* and *method of adjacent equilibria*; see [441,777]. To apply this criterion we look at *admissible static perturbations of an equilibrium position*.¹ These perturbations generate adjacent states or configurations, which are not necessarily in equilibrium.

Stability is assessed by comparing the potential energy of these adjacent configurations to that of the equilibrium position. If all adjacent states have a higher potential energy, the equilibrium is stable. If at least one state has a lower (equal) potential energy the equilibrium is unstable (neutrally stable). This comparison can be expressed in terms of the second variation of the potential energy and thus reduced to assessing the *positive definite character of the tangent stiffness matrix*.

Although stability is inherently dynamic, mass or damping information is not needed in the static criterion. This is a key reason for its popularity. But its derivation makes it strictly applicable only to *conservatively loaded* systems since a load potential function is assumed to exist.

The dynamic criterion is succinctly covered in the last two chapters of the book. It is applicable to both conservative and nonconservative systems, as is evident from (29.1). Its key disadvantage is that additional structural properties, such as mass, are required.

§29.3.1. The Tangent Stiffness Test

The ensuing material expands on that of §5.2. The stability of conservative systems can be assessed by looking at the spectrum² of the tangent stiffness matrix \mathbf{K} at an *equilibrium* configuration. Let μ_i denote the i^{th} eigenvalue of \mathbf{K} . The set of μ_i 's are obtained from the solution of the algebraic eigenproblem

$$\mathbf{K} \mathbf{z}_i = \mu_i \mathbf{z}_i. \quad (29.2)$$

Since \mathbf{K} is *real symmetric*,³ all of its eigenvalues are *real*. As a result, we can administer the following spectral test:

Case	Condition	The equilibrium configuration is
(I)	All $\mu_i > 0$	<i>Strongly stable</i>
(II)	All $\mu_i \geq 0$ and at least one $\mu_i = 0$	<i>Neutrally stable</i>
(III)	At least one $\mu_i < 0$	<i>Unstable</i>

(29.3)

Mathematically, \mathbf{K} is said to be *positive definite*, *nonnegative* and *indefinite* in cases (I), (II) and (III), respectively. The foregoing classification constitutes the *tangent stiffness test*, or TST, to assess stability. It should not be confused with the *singular stiffness test*, or SST, described in §29.3.2. Carrying out the SST involves solving a (generally nonlinear) eigenproblem whereas doing the TST requires only a matrix factorization; see Remark 29.2.

¹ “Admissible” in the sense of the Principle of Virtual Work: variations of the state parameters that are consistent with the essential boundary conditions (kinematic constraints in structural mechanics). Cf. Table 1.2.

² The spectrum of a matrix is the set of its eigenvalues.

³ Because $\mathbf{K} = \partial^2 \Pi / \partial \mathbf{u} \partial \mathbf{u}$ is the Hessian of the total potential energy Π . Cf. §5.2.2.

Remark 29.1. The TST (29.3) is based on the Lagrange-Dirichlet theorem: “The equilibrium of a system subjected only to conservative and dissipative forces is stable if its total potential energy is continuous and has a strict minimum (that is, its Hessian is positive definite), with respect to the state variables.” In the present treatment, dissipative (damping) forces are ignored. The theorem was enunciated by Lagrange in his *Mécanique Analytique* [453] and rigorously proven 50 years later by Dirichlet [193]. For the proof, see e.g., [65, Section 3.6], in which geometric arguments in phase space are used.

Remark 29.2. If \mathbf{K} is known at a given λ , an explicit solution of the eigenproblem (29.2) is not necessary for assessing stability. It is sufficient to factor \mathbf{K} as

$$\mathbf{K} = \mathbf{L}\mathbf{D}\mathbf{L}^T, \quad (29.4)$$

in which \mathbf{L} is unit lower triangular and \mathbf{D} is diagonal. The number of negative eigenvalues of \mathbf{K} is equal to the number of negative diagonal elements (“pivots”) of \mathbf{D} . Matrix factorization is considerably cheaper than carrying out a complete eigenanalysis because sparseness can be exploited more effectively.

§29.3.2. The Singular Stiffness Test

The structural engineer is especially interested in the behavior as the stage control parameter λ is varied. Consequently

$$\mathbf{K} = \mathbf{K}(\lambda). \quad (29.5)$$

Given this dependence, a key information is the *transition* from stability to instability at the value of λ closest to stage start, which is usually taken as $\lambda = 0$. This is the *first critical point* or FCP; see Table 1.2. The value of λ at the FCP is called the *critical value* of λ , denoted as λ_{cr} .

If the entries of \mathbf{K} depend continuously on λ the eigenvalues of \mathbf{K} also depend continuously⁴ on λ , although the dependence is not necessarily continuously differentiable. It follows that transition from strong stability — case (I) — to instability — case (III) — has to go through case (II), *i.e.* a zero eigenvalue. Thus a *necessary* condition is that \mathbf{K} be singular, that is

$$\det \mathbf{K}(\lambda_{cr}) = 0, \quad (29.6)$$

or, equivalently,

$$\mathbf{K}(\mathbf{u}_{cr}, \lambda_{cr}) \mathbf{z}_{cr} = \mathbf{0}, \quad (29.7)$$

in which $\mathbf{z}_{cr} \neq \mathbf{0}$ is the mode shape introduced in Chapter 5, where it was called a *null eigenvector*. If the FCP is of bifurcation type, eigenvector \mathbf{z}_{cr} is called a *buckling mode*. Either (29.6) or (29.7) expresses the *singular stiffness test*, or SST, for finding the transition of stability to instability.

Remark 29.3. The condition (29.6) is *not sufficient* for concluding that a system that is stable for $\lambda < \lambda_{cr}$ will go unstable as λ exceeds λ_{cr} . A counterexample is provided by the stable-symmetric bifurcation point discussed later. The Euler Column furnishes a well known instance. At such points (29.6) holds implying neutral stability. However, stability is not lost as the bifurcation is traversed and the structure moves to a secondary path. Of course displacements may become so large that the structure is practically rendered useless, or elasticity may be lost.

⁴ Continuous dependence of eigenvalues on the entries is guaranteed by the perturbation theory for symmetric and Hermitian matrices [733]. This continuous dependence does not hold, however, for eigenvectors.

§29.4. Linearized Prebuckling

The SST eigenproblem (29.7) can be drastically simplified by assuming that all displacements prior to the first critical point (FCP) can be neglected. This leads to the so-called Linearized Prebuckling (LPB) method of stability analysis. As a consequence, (29.7) becomes a *linear eigenproblem*, which can be effectively solved by standard linear algebra packages.⁵

Why “prebuckling” and not “presnapping”? We will prove in Chapter 31 that an important LPB limitation is its *inability to detect limit points*. Thus, in structural applications it can only predict buckling. PB modeling assumptions are discussed in more depth in §31.4. In this section we discuss the formulation of the LPB eigenproblem and illustrate it on simple problems.

§29.4.1. The LPB Eigensystem

The key consequences from the LPB assumption, which are studied in detail in following chapters, can be summarized as follows. Recall from the FEM formulation chapters that the tangent stiffness matrix can be decomposed as the sum of material and geometric stiffness matrices:

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G. \quad (29.8)$$

Then the LPB assumptions lead to the following simplifications:

(1) The material stiffness is the linear stiffness evaluated at the reference configuration:

$$\mathbf{K}_M = \mathbf{K}_0. \quad (29.9)$$

(2) The geometric stiffness is linearly dependent on the control parameter λ :

$$\mathbf{K}_G = \lambda \mathbf{K}_1. \quad (29.10)$$

in which \mathbf{K}_1 is a constant matrix that is also evaluated at the reference configuration.

The SST requires that \mathbf{K} be singular. From (29.7) we get

$$\mathbf{K} \mathbf{z}_i = (\mathbf{K}_0 + \lambda_i \mathbf{K}_1) \mathbf{z}_i = \mathbf{0}. \quad (29.11)$$

This is called the *LPB stability eigenproblem*. The eigenvalue λ_i closest to the stage start $\lambda = 0$ is the critical control parameter λ_{cr} or, for a single-stage problem, the critical load factor.

Since \mathbf{K}_0 and \mathbf{K}_1 are real, constant and symmetric, the eigenproblem (29.11) befits the generalized symmetric algebraic eigenproblem of linear algebra:

$$\mathbf{A} \mathbf{x}_i = \lambda_i \mathbf{B} \mathbf{x}_i, \quad (29.12)$$

in which both matrices $\mathbf{A} \equiv \mathbf{K}_0$ and $\mathbf{B} \equiv -\mathbf{K}_1$ are real symmetric, and $\mathbf{x} \equiv \mathbf{z}$ are the buckling mode eigenvectors. If (as usual) the material stiffness \mathbf{K}_0 is positive definite, eigensystem theory says that all eigenvalues of (29.12) are real. We cannot in general make statements, however, about the *sign* of those eigenvalues. That will depend on the physics of the problem as well as on the sign conventions chosen for λ . The solution of (29.12) is discussed in §29.4.3.

⁵ Such packages are standard components of higher order scientific languages such as *Matlab* and *Mathematica*.

§29.4.2. Summary of LPB Steps

The LPB eigenanalysis steps are summarized below. Those are illustrated with examples in §29.5.

1. *Prebuckling Analysis.* It is assumed that the external loading is separable and proportional: $\mathbf{f} = \lambda \mathbf{q}$, in which the reference load \mathbf{q} is constant. Assemble the linear stiffness \mathbf{K}_0 in the reference configuration and solve the linear static problem for $\lambda = 1$:

$$\mathbf{K}_0 \mathbf{u}_0 = \mathbf{q}. \quad (29.13)$$

From the solution \mathbf{u}_0 obtain the internal force (stress) distribution for use in Step 2. Note: In *statically determinate* structures the internal forces and stresses may be obtained directly from equilibrium so the linear analysis (29.13) may be skipped. Nonetheless \mathbf{K}_0 is still necessary for Step 3.

2. *Eigensystem Set Up.* The stress distribution obtained from Step 1 is taken as the *initial stress* \mathbf{s}_0 in the reference configuration. Using this information, assemble the reference geometric stiffness \mathbf{K}_1 so that the geometric stiffness is $\mathbf{K}_G = \lambda \mathbf{K}_1$.
3. *Eigensystem Analysis.* Solve the LPB stability eigenproblem

$$(\mathbf{K}_0 + \lambda_i \mathbf{K}_1) \mathbf{z}_i = \mathbf{0}, \quad \text{or} \quad \mathbf{K}_0 \mathbf{z}_i = -\lambda_i \mathbf{K}_1 \mathbf{z}_i, \quad (29.14)$$

The eigenvalue λ_i closest to zero is the critical load factor, while the associated eigenvector \mathbf{z}_i gives the corresponding buckling mode.

One important reminder: the kinematic boundary conditions (BC) for Steps 1 and 3 are not necessarily the same. A typical case is illustrated in Figure 31.3. In Step 1 axial displacements (in the y direction) must be allowed so the column can develop axial strains and stresses through contraction. In Step 3 (eigensolution) those axial displacements are suppressed, since buckling will trigger only lateral deflections. In practice the BC for Step 3 can be often updated as a result of examining the eigensolution, especially the presence of infinite eigenvalues caused by the presence of null rows and columns in \mathbf{K}_1 .

§29.4.3. Stability Eigenproblem Solution

In production FEM codes, the LPB stability eigenproblem (29.14) is generally treated with special solution techniques that take full advantage of the sparsity of both \mathbf{K}_0 and \mathbf{K}_1 ; for example subspace iteration or Lanczos-Arnoldi methods [573,688]. An important consideration to attain high efficiency is that often only one eigenvalue/vector pair is of interest.⁶

For small systems an expedient solution method consists of reducing it to *canonical form* by premultiplying both sides by the inverse of \mathbf{K}_0 . This is possible if \mathbf{K}_0 is nonsingular, which means that the $\lambda = 0$ configuration is not a critical one.⁷ Renaming for convenience $\mathbf{A} = \mathbf{K}_0^{-1} \mathbf{K}_1$ and

⁶ In special circumstances more than one eigensolution pair may be required. For example, if there is a mix of positive and negative eigenvalues; or if there are multiple or closely-separated eigenvalues.

⁷ If \mathbf{K}_0 is singular, either the reference configuration is a critical point, or rigid body modes have not been fully suppressed. Additional support conditions are necessary in the latter case.

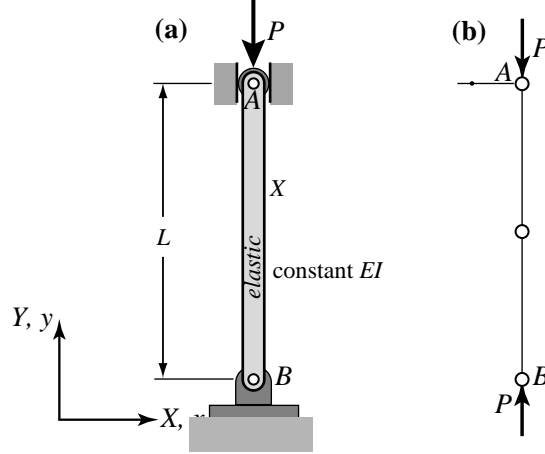


FIGURE 29.1. LPB example: extensional-spring-propped rigid hinged rigid cantilever (ESPHRC) column treated by FEM in a TL framework. Infinitesimal displacements u_{X1} and u_{Y1} are highly exaggerated in the Figure for visibility.

$\mu_i = -1/\lambda_i$ one gets

$$\mathbf{A} \mathbf{z}_i = \mu_i \mathbf{z}_i \quad (29.15)$$

This is a standard algebraic eigenproblem, which can be solved by standard library routines for the eigenvalues μ_i and eigenvectors \mathbf{z}_i . For example, `Eigensystem` in *Mathematica* or `eig` in *Matlab*. The μ_i farthest away from zero gives the $\lambda_i = -1/\mu_i$ closest to zero.

Disadvantages of this reduction are that \mathbf{A} is unsymmetric even if \mathbf{K}_0 and \mathbf{K}_1 are, and that the original sparseness is lost. There are more complicated reduction methods, based on the Cholesky decomposition, that preserve symmetry if at least one of the matrices is nonnegative definite. These may be studied in textbooks covering linear algebra; for example [323]. But — as previously noted — in production FEM codes the generalized eigenproblem (29.11) is treated directly.

§29.5. LPB Analysis Examples

§29.5.1. ESPHRC Column

We retake the problem of the extensional-spring-propped hinged rigid cantilevered column treated in §28.5.3 and §28.5.4, but now in finite element plumage. A Total Lagrangian (TL) FEM model is shown in Figure 29.1. The column is modeled as a 2-node TL bar element connected to a linear extensional spring. Two major differences with respect to the equilibrium model shown in Figure 28.6 should be noted:

1. The column is now elastic, as a rigid member cannot be handled as a standard finite element.
2. The system has two degrees of freedom (DOF): the displacements u_{X1} and u_{Y1} of node 1.

The internal force in the column is evidently $F = -P$ and the initial axial stress in \mathcal{C}_0 is $s_0 = F/A_0 = -P/A_0 = -\lambda P_{ref}/A_0$. Consequently the first LPB step can be skipped, and the TL geometric stiffness (?) set up by inspection. Denote by $k_C = EA_0/L_0$ the equivalent linear stiffness of the column. Upon suppressing the displacement of node 2 (hinge) the LPB eigensystem is

$$\begin{bmatrix} k & 0 \\ 0 & EA_0/L_0 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix} = \frac{P}{L_0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix} = \lambda \frac{P_{ref}}{L_0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix}. \quad (29.16)$$

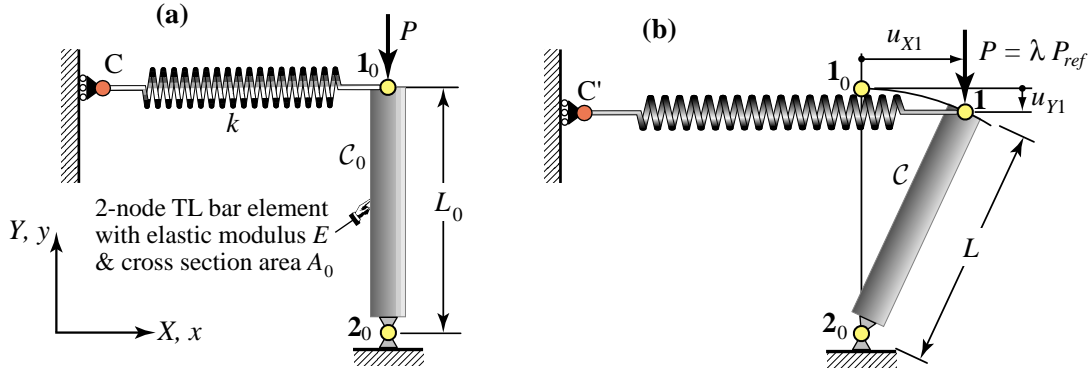


FIGURE 29.2. LPB example: Torsional-spring-propped rigid hinged rigid cantilever (ESPHRC) column treated by FEM in a TL framework. Infinitesimal displacements u_{X1} and u_{Y1} are highly exaggerated in the Figure for visibility.

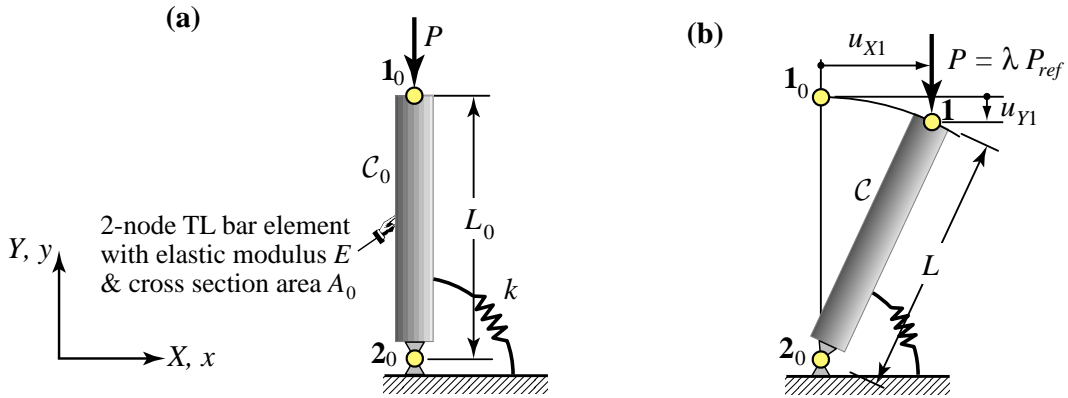


FIGURE 29.3. TSPHRC column treated by a FEM discretization.

Taking $P_{ref} = 1$ for convenience, the two critical load factors are

$$\lambda_{cr1} = k L_0, \quad \lambda_{cr2} = E A_0. \quad (29.17)$$

The eigenvectors are $[1 \ 0]^T$ and $[0 \ 1]^T$. Therefore λ_{cr1} is associated with the tilted column buckling, whereas λ_{cr2} has no physical significance as it pertains to a load that would reduce the column length to zero. To recover the rigid column case, make $E A_0 \rightarrow \infty$ whereupon $\lambda_{cr1} = k L_0 = k L$ and $\lambda_{cr2} \rightarrow \infty$. The first one agrees with the results of §28.5.4.

Another way to recover the rigid column is to apply the multifreedom constraint u_{Y1} , using the master-slave matrix transformation

$$\begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u_{X1}]. \quad (29.18)$$

Applying this as congruential transformation to (29.19) eliminates u_{Y1} and leaves only $\lambda_{cr1} = \lambda_{cr} = k L$.

§29.5.2. TSPHRC Column

Here we tried to redo the problem of the torsional-spring-propped hinged rigid cantilevered column treated in §28.5.1 and §28.5.2, with a finite element TL model, which is shown in Figure 29.2. The column is modeled as a 2-node TL bar element connected to a torsional spring.

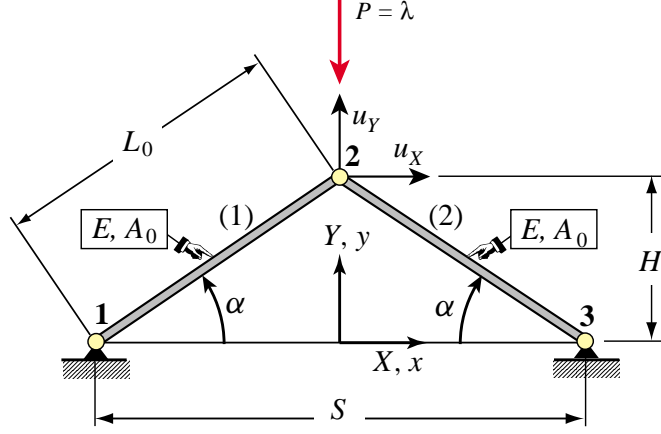


FIGURE 29.4. FEM-discretized Mises truss. A reproduction of Figure ? with only one change.

The internal force in the column is evidently $F = -P$ and the initial axial stress in C_0 is $s_0 = F/A_0 = -P/A_0 = -\lambda P_{ref}/A_0$. Consequently the first LPB step can be skipped, and the TL geometric stiffness (?) set up by inspection. Denote by $k_C = EA_0/L_0$ the equivalent linear stiffness of the column. Upon suppressing the displacement of node 2 (hinge) the LPB eigensystem is

$$\begin{bmatrix} k & 0 \\ 0 & EA_0/L_0 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix} = \frac{P}{L_0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix} = \lambda \frac{P_{ref}}{L_0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \end{bmatrix}. \quad (29.19)$$

Taking $P_{ref} = 1$ for convenience, the two critical load factors are

$$\lambda_{cr1} = k L_0, \quad \lambda_{cr2} = EA_0. \quad (29.20)$$

§29.5.3. The Mises Truss

Consider the Mises truss studied previously in the context of geometrically nonlinear analysis. It is defined in Figure 29.4, which is a reproduction of Figure ?. The only change is that $P = \lambda$ is taken as positive if the crown load compresses the truss. The FEM discretization consists of two bar members and three nodes. The initial forces in the bars are easily obtained from statics: $-P/(2A_0 \sin \alpha)$. Upon applying the hinged support conditions at nodes 1 and 3, a simple hand computation gives the LPB eigenproblem as

$$\frac{EA_0}{L_0^3} \begin{bmatrix} S^2/2 & 0 \\ 0 & 2H^2 \end{bmatrix} \begin{bmatrix} u_{X2} \\ u_{Y2} \end{bmatrix} = \frac{\lambda}{L_0 \sin \alpha} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{X2} \\ u_{Y2} \end{bmatrix}. \quad (29.21)$$

in which $L_0 = \frac{1}{2}\sqrt{H^2 + \frac{1}{4}S^2}$ is the initial length of both bars. Accordingly the two LPB eigenvalues are

$$\lambda_1 = \frac{EA_0 H^2 \sin \alpha}{L_0^2} = \frac{EA_0}{L_0} \sin^3 \alpha, \quad \lambda_2 = \frac{EA_0 S^2 \sin \alpha}{4L_0^2} = \frac{EA_0}{L_0} \sin \alpha \cos^2 \alpha. \quad (29.22)$$

If $\alpha > 45^\circ$, λ_2 is critical; its eigenvector is $u_{X2} = 1, u_{Y2} = 0$ (truss buckles horizontally). If $\alpha < 45^\circ$, λ_1 is critical; its eigenvector is $u_{X2} = 0, u_{Y2} = 1$ (truss buckles vertically). If $\alpha = 45^\circ$, the eigenvalues coalesce. The LPB predictions are nonsensical for this structure, since deformations prior to reaching a critical point cannot be neglected.

Notes and Bibliography

The linearized stability analysis described here dominates pre-1970 publications, since fully nonlinear methods require computer help and FEM discretizations to be practical. For example, the classical Timoshenko-Gere textbook [775] barely mentions nonlinear and postbuckling analysis in the second edition and primarily covers only LPB, which is just called “buckling.” The term “Linearized Prebuckling” was coined by Brush and Almroth [115].

The LPB division into three steps presented here is not mentioned elsewhere.

Homework Exercises for Chapter 29

Linearized Prebuckling: Formulation

EXERCISE 29.1 [A:15] Find the buckling mode (null eigenvector) \mathbf{z} (normalized to unit length) for the example problem of §28.6, and verify the orthogonality condition $\mathbf{z}^T \mathbf{q} = 0$.

EXERCISE 29.2 [A:15] Find the buckling load for the two-bar problem if bar (2) forms an angle $0 \leq \varphi < 90^\circ$ with the x axis. Assume still that the system displaces only on the $x - y$ plane and that $k^{(2)} \ll k^{(1)}$ so that the stress in bar (2) can be neglected in forming the geometric stiffness. Determine \mathbf{z} and verify orthogonality.

EXERCISE 29.3 [A:20] Suppose the load of the two-bar example problem of §28.6 depends on the vertical displacement of point 2 as $p = -\lambda c u_{Y2}^2$, where c is a constant with dimensions of stress. Show that even if prebuckling deformations are neglected, the singular stiffness test leads to a nonlinear eigenvalue problem.

EXERCISE 29.4 [A:25] The “Euler column” shown in Figure E29.1 is modelled by *one* 2-node Euler-Bernoulli beam column element along its length:

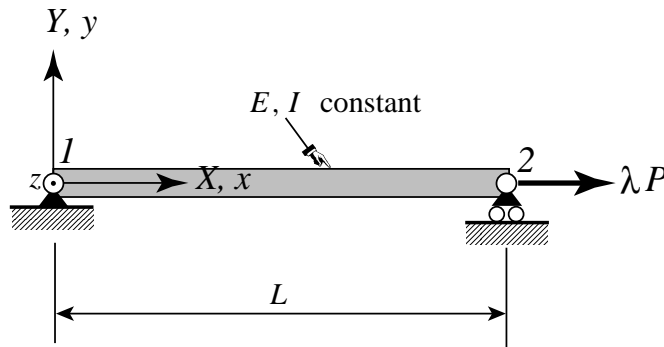


FIGURE E29.1. One-element model of Euler column.

The state variables are the nodal displacements degrees of freedom arranged as

$$\mathbf{u} = \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ \theta_{z1} \\ u_{X2} \\ u_{Y2} \\ \theta_{z2} \end{Bmatrix} \quad (\text{E29.1})$$

where θ_{z1} and θ_{z2} are (to first order) the end rotations, positive counterclockwise about z .

The linear material matrix in the reference (undeformed) configuration is

$$\mathbf{K}_0 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (\text{E29.2})$$

[symm]

in which E is the elastic modulus, L the element length, A the cross section area, and I the moment of inertia of the cross section about the z neutral axis.

The exactly-integrated geometric stiffness at the reference configuration is⁸

$$\mathbf{K}_G = \lambda \mathbf{K}_1, \quad \mathbf{K}_1 = \frac{P}{30L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 36 & 3L & 0 & -36 & 3L \\ & & 4L^2 & 0 & -3L & -L^2 \\ & & & 0 & 0 & 0 \\ & \text{symm} & & & 36 & -3L \\ & & & & & 4L^2 \end{bmatrix} \quad (\text{E29.3})$$

where N is the axial force in the element (here obviously equal to the applied force λP because the structure is statically determinate.)

For this problem:

- (a) Check that \mathbf{K}_0 and \mathbf{K}_1 satisfy translational infinitesimal rigid body motion conditions $u_X \equiv 1$ and $u_Y \equiv 1$ if the six degrees of freedom are left unconstrained. (Convert those modes to node displacements, then premultiply by the stiffness matrices.)

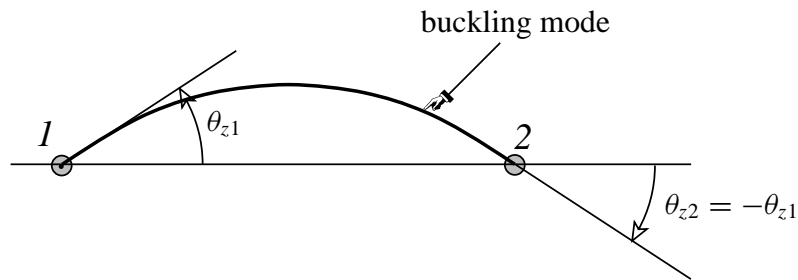


FIGURE E29.2. Symmetric buckling of one-FE model of Euler column.

- (b) Set up the linearized prebuckling eigenproblem

$$(\mathbf{K}_0 + \lambda \mathbf{K}_1) \mathbf{z} = \mathbf{0} \quad (\text{E29.4})$$

Apply the support end conditions to remove u_{X1} , u_{Y1} and u_{Y2} as degrees of freedom.

- (c) Justify that freedom u_{X2} can be isolated from the eigenproblem, and proceed to drop it to reduce the eigenproblem to 2×2 .
- (d) Reduce the 2×2 eigenproblem to a scalar one for the symmetric buckling mode sketched in Figure E29.2. by setting $\theta_{z1} = -\theta_{z2}$ (see Figure), and get the first critical load parameter λ_1 .
- (e) Repeat (e) for the antisymmetric buckling mode sketched in Figure E29.3 by setting $\theta_{z1} = \theta_{z2}$ (see Figure) and obtain the second critical load parameter λ_2 .
- (f) Compare the results of (d)–(e) to the exact critical load values

$$P_1^E = -\pi^2 \frac{EI}{L^2}, \quad P_2^E = -4\pi^2 \frac{EI}{L^2}, \quad (\text{E29.5})$$

⁸ See e.g., Przemieniecki's book [630].

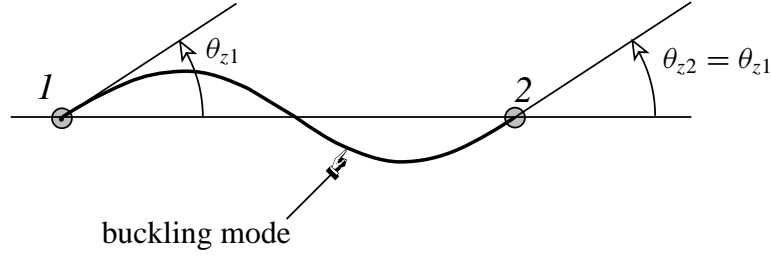


FIGURE E29.3. Antisymmetric buckling of one-FE model of Euler column.

The first one was determined by Euler in 1744 and therefore is called the *Euler critical load*. For P_1^E the FEM result should be within 25%, which is good for one element.

- (g) Repeat the calculations of λ_1 and λ_2 with the following reduced-integration geometric stiffness matrices⁶

$$\mathbf{K}_1^* = \frac{P}{8L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 9 & 3L & 0 & -9 & 3L \\ & & L^2 & 0 & -3L & -L^2 \\ & & & 0 & 0 & 0 \\ & & & & 9 & -3L \\ & \text{symm} & & & & L^2 \end{bmatrix} \quad (\text{E29.6})$$

$$\mathbf{K}_1^{**} = \frac{P}{24L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 24 & 0 & 0 & -24 & 0 \\ & & 2L^2 & 0 & 0 & -2L^2 \\ & & & 0 & 0 & 0 \\ & & & & 24 & 0 \\ & \text{symm} & & & & 2L^2 \end{bmatrix} \quad (\text{E29.7})$$

and comment on the relative accuracy obtained against the exact values.

- (h) Repeat the calculations of steps (b) through (e) for a *two* equal-element discretization. Verify that the symmetric-mode buckling load is now $-10EI/L^2$, which is (surprisingly) close to Euler's value $P_1^E = -\pi^2 EI/L^2$.

EXERCISE 29.5 [A:25] This is identical in all respects to Exercise 29.4, except that the 2-node Timoshenko beam model is used, with Mac Neals's RBF correction for the material stiffness matrix. The net result is that \mathbf{K}_0 is *identical* to (E28.2) but \mathbf{K}_1 is different. In fact $\lambda \mathbf{K}_1$ is given by Equation (9.45) where $V = 0$ and $N = \lambda P$. Note: don't be surprised if the one-element results are poor when compared to the analytical buckling load.

EXERCISE 29.6 [A/C: 20] The column shown in Figure E29.4 consists of 3 rigid bars of equal length L connected by hinges and stabilized by two lateral springs of linear stiffness k . The applied axial load is λP , where $\lambda > 0$ means compression. Compute the two buckling loads $\lambda_1 P$ and $\lambda_2 P$ in terms of k and L , assuming the LPB model of infinitesimal displacements from the initial state. Show that one load corresponds to a symmetric buckling mode and the other to an antisymmetric buckling mode, and find which one is critical.

Hint. The two degrees of freedom are the small lateral displacements u_1 and u_2 of hinges 1 and 2, where $u_1 \ll L$, $u_2 \ll L$. Write the total potential energy as

$$\Pi = U - W, \quad U = \frac{1}{2}ku_1^2 + \frac{1}{2}ku_2^2, \quad W = \frac{\lambda P}{2L}(u_1^2 + (u_1 - u_2)^2 + u_2^2). \quad (\text{E29.8})$$

⁶ These matrices are obtained by one-point and two-point Gauss integration, respectively, whereas the \mathbf{K}_1 of (E28.3) is obtained by either 3-point Gauss or analytical integration.

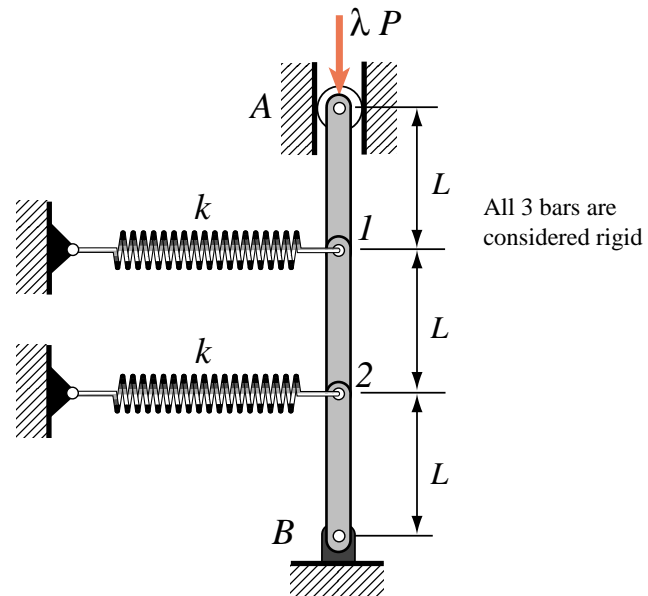


FIGURE E29.4. Buckling of a segmented-hinged column propped by two springs.

Explain where the expression of W comes from. Once Π is in hand, it is smooth sailing.