# 29

# Linearized Prebuckling: Formulation

# Chapter Overview

**This Chapter covers the simplest practical method for analyzing static structural stability: Linearized Prebuckling, acronymed as LPB.**

**Achieving that simplicity requires making several behavioral assumptions. These are discussed in more detail later.**

**If those assumptions are satisfied, or at least acceptable, LPB leads to a linear eigenvalue problem that involve constant matrices. This eigenproblem can be processed by well tested linear algebra packages.**

**Its solution provides critical loads as eigenvalues and buckling mode shapes as eigenvectors.**

# Key Advantage of LPB

**The need for carrying out a full nonlinear analysis for finding critical points is completely bypassed.**

**That kind of analysis can be time consuming in terms of human effort as well as computationally demanding - getting a response may take days or weeks on supercomputers.**

**Furthermore, it typically requires detailed information about the structure, which may not be available during preliminary design stages.**

# Key Limitations

**The gain in simplicity and speed is balanced by modeling limitations:**

**LPB can only provide information on bifurcation points located in an undeflected primary path, since deformations prior to buckling are neglected.**

**Two consequences:**

**LPB cannot detect limit points**
**All information on postbuckling behavior is lost**

**(And of course, dynamic stability is completely beyond the scope of LPB)**

No

## Main Application:   Preliminary Design

**LPB predictions are often sufficient for preliminary design of many structures, particularly those in Civil Engineering (design of typical CE structures is governed by building co**

**It is often possible to use simple FE models during this stag e.g. 1D elements instead of 2D or 3D.**

**Ordinary safety factors may be sufficient to cover shortcomings.  And more realistic verification analyses ma be performed if necessary after a design is firmly establish**

# Recall the Two Ways of Setting Up Static Stability Equations

## Equilibrium Method

**use FBD in perturbed equilibrium configuration, look for nontrivial solutions as function of $\lambda$**

## Energy Method

**set up total potential energy of system, analyze whether its Hessian is positive definite (S), nonnegative definite (N), or indefinite (U), as function of $\lambda$**

**LPB is conveniently done using the Energy Method within a FEM/DSM framework**

# Setting Up the LPB Equations in FEM

**To set up LPB in a FEM/DSM framework, we combine two related mathematical tools:**

**The Singular Stiffness Test:**

**The tangent stiffness is singular at a critical point**

**The Tangent Stiffness Test:**

**The tangent stiffness spectrum characterizes stability**

**with the simplfying kinematic assumption**

**Deformations prior to buckling can be neglected:**

**We can use the reference configuration geometry**

# The Tangent Stiffness Test (1)

**Consider a structural system that satisfies the conditions for static stability analysis to be valid: conservative loading and linearly elastic material.**

**As noted in Chapter 5, the transition of such system from stability to instability can only occur at critical points.**

**At such points the tangent stiffness matrix K becomes singular. But what happens at regular points? We must look at the full spectrum of K (next slide)**

# The Tangent Stiffness Test (2)

**Let K be the tangent stiffness at the equilibrium configuration of a conservative system to be tested for stability. Consider the algebraic eigenproblem**

$$\boxed{\mathbf{K}\,\mathbf{z}_i = \mu_i\,\mathbf{z}_i}$$

**Since K is real symmetric all of its eigenvalues are real. As a result we can administer the following test:**

| Case | Condition | The configuration is |
|---|---|---|
| (I) | All $\mu_i > 0$ | Strongly stable |
| (II) | All $\mu_i$ are nonnegative and at least one is zero | Neutrally stable |
| (III) | At least one $\mu_i < 0$ | Unstable |

**Mathematically: K is said to be positive definite, nonnegative and indefinite in cases (I), (II) and (III), respectively**

# The Tangent Stiffness Test (3)

**In practice K is a function of the stage control parameter** $\lambda$ **as one moves over the primary equilibrium path:**

$$\mathbf{K} = \mathbf{K}(\lambda)$$

**Assume that the structure is strongly stable, meaning case (I), for sufficiently small values of** $\lambda$**, in particular** $\lambda = 0$.
**Transition to the unstable case (III) must occur by going through neutral stability, which is case (II). Let** $\lambda_{cr}$ **denote the first value for which at least one eigenvalue becomes zero. Then** $\mathbf{K}_{cr} = \mathbf{K}(\lambda_{cr})$ **is singular, or equivalently**

$$\det\ \mathbf{K}(\lambda_{cr}) = 0$$

**This is the singular stiffness test in terms of the control parameter** $\lambda$

# Tangent Stiffness Test (4)

**If K is known at a given $\lambda$, an explicit solution of the eigenproblem K $\mathbf{z}_i = \mu_i \mathbf{z}_i$ is not necessary for assessing stability. It is sufficient to factor K as**

$$\boxed{\mathbf{K} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T}$$

**in which L is unit lower triangular and D is diagonal.**

**The number of negative eigenvalues of K is equal to the number of negative diagonal elements ("pivots" of D).**

**Matrix factorization is considerably cheaper than carrying out a complete eigenanalysis because sparseness can be exploted more effectively.**

# The Singular Stiffness Test (1)

**The structural engineer is especially interested in what happens as the control parameter $\lambda$ is varied. Consequently, along an equilibrium path**

$$\mathbf{K} = \mathbf{K}(\lambda)$$

**The key information is the transition from stability to instability at a $\lambda$ closest to analysis start, which is usually taken to be $\lambda = 0$.**

**This is the first critical point (FCP). The value of $\lambda$ at the FCP is called the critical value of $\lambda$, and is denoted by $\lambda_{cr}$**

# Clarification of a Source of Confusion

**In separable problems K is only a function of u: K(u). In non-separable problems, $K = K(u,\lambda)$. Why then we say that**

$$\boxed{K = K(\lambda)\ \ ?}$$

**Remember that we are moving along an equilibrium path, which is**

$$\boxed{r(u,\lambda) = 0}$$

**That means $u = u(\lambda)$. Replacing the u in K(u) or $K(u,\lambda)$ we end up with $K(\lambda)$.**

# The Singular Stiffness Test (2)

**If the entries of K depend continuously on $\lambda$, the eigenvalues of K also depend continuously on $\lambda$, although the dependence is not necessarily continuously differentiable. It follows that transition form strong stability: case (I), to instability: case (III) has to go through case (II); that is, a zero eigenvalue. Thus a necessary condition is that K be singular:**

$$\det \mathbf{K}\,(\lambda_{cr}) = 0$$

**or, equivalently**

$$\mathbf{K}(\mathbf{u}_{cr}, \lambda_{cr})\, \mathbf{z}_{cr} = 0$$

**in which $\mathbf{z}_{cr}$ was introduced in Chapter 5, where it was called a null eigenvector. If the FCP is of bifurcation type, that eigenvector is called a buckling mode.**

**Why "necessary" but not "sufficient"? -> Next slide.**

# Singular Stiffness Test (3)

**As noted in the last slide, if**

$$\det \mathbf{K}(\lambda_{cr}) = 0$$

**the neutrally stable structure: case (II), is at a critical point.**
**[The converse is not necessarily true: the structure can be at a critical point but be unstable; it depends on the whole spectrum].**

**Because of the continuous dependence of eigenvalues of K on the parameter $\lambda$, transsition from (I) to (III) must go through (II). This can be enunciated as the transition property:**

**Transition from stability to instability always occurs at a critical point**

**[This useful property does not hold in dynamic stability.]**

# Singular Stiffness Test (4)

**Note that**

$$\det \, \mathbf{K}(\lambda) = 0$$

**represents a nonlinear algebraic eigenproblem because K depends on u, which in turn is a function of $\lambda$ through the nonlinear residual equilibrium equation. Thus solving the foregoing equation for the FCP in general requires tracing the primary response. This can be a very expensive task.**

**A significant simplification occurs if we can ignore the change in geometry before the critical configuration is reached. Through partial linearization of K this leads to the Linearized Prebuckling (LPB) eigenproblem**

# Linearized Prebuckling (1)

**The modeling assumptions that are tacitly or explicitly made in LPB are discussed in detail in the next Chapter, as well as the practical limitations that result from those assumptions.**

**In the present Chapter we discuss the formulation of the LPB eigenproblem and illustrate those techniques with simple problems.**

# Linearized Prebuckling (2)

**Recall that K can be split into material and geometric stiffness:**

$$\mathbf{K} = \mathbf{K}_M + \mathbf{K}_G$$

**Since geometric changes prior to the critical state are neglected, we can assume that**

$$\mathbf{K}_M = \mathbf{K}_0$$

**which is the material stiffness matrix evaluated at the reference state** $\lambda = 0$. **Further**

$$\mathbf{K}_G = \lambda\ \mathbf{K}_1$$

**in which** $\mathbf{K}_1$ **is the geometric stiffness for** $\lambda = 1$, **also evaluated at the reference state. This** $\mathbf{K}_1$ **is called the reference geometric stiffness, or unit geometric stiffness.**

# Linearized Prebuckling (3)

**Recall the Tangent Stiffness Test eigenproblem**

$$\boxed{\mathbf{K}\,\mathbf{z}_i = \mu_i\,\mathbf{z}_i}$$

**Insert the split K = K$_M$ + K$_G$ = K$_0$ + $\lambda$ K$_1$ to get the LPB eigenproblem**

$$\boxed{\mathbf{K}\,\mathbf{z}_i = (\mathbf{K}_0 + \lambda_i \mathbf{K}_1)\,\mathbf{z}_i = \mathbf{0}}$$

**Since K$_0$ and K$_1$ are constant and real symmetric matrices, this is a generalized symmetric algebraic eigenproblem that can be solved with standard software packages, for example thosde provided in *Matlab*.**

# Summary of LPB Steps

1. ***Prebuckling Analysis***. It is assumed that the external loading is separable and proportional: $\mathbf{f} = \lambda\mathbf{q}$, in which the reference load $\mathbf{q}$ is constant. Assemble the linear stiffness $\mathbf{K}_0$ in the reference configuration and solve the linear static problem for $\lambda = 1$:

$$\boxed{\mathbf{K}_0\,\mathbf{u}_0 = \mathbf{q}}$$

   From the solution $\mathbf{u}_0$ obtain the internal force (stress) distribution for use in Step 2. Note: In *statically determinate* structures the internal forces and stresses may be obtained directly from equilibrium so the linear analysis (29.13) may be skipped. Nonetheless $\mathbf{K}_0$ is still necessary for Step 3.

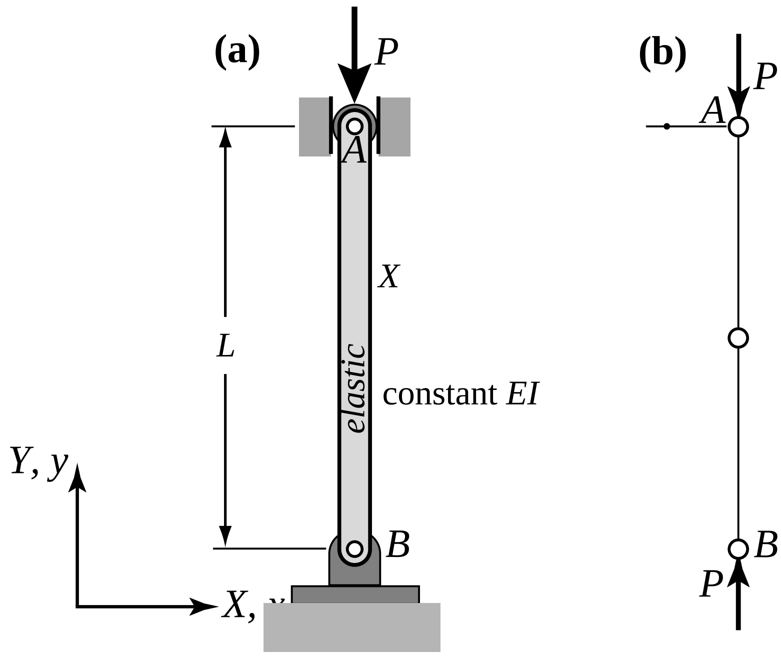
2. ***Eigensystem Set Up***. The stress distribution obtained from Step 1 is taken as the *initial stress* $\mathbf{s}_0$ in the reference configuration. Using this information, assemble the reference geometric stiffness $\mathbf{K}_1$ so that the geometric stiffness is $\mathbf{K}_G = \lambda\,\mathbf{K}_1$.

3. ***Eigensystem Analysis***. Solve the LPB stability eigenproblem

$$\boxed{(\mathbf{K}_0 + \lambda_i\,\mathbf{K}_1)\,\mathbf{z}_i = \mathbf{0} \qquad \text{or} \qquad \mathbf{K}_0\,\mathbf{z}_i = -\lambda_i\,\mathbf{K}_1\,\mathbf{z}_i}$$

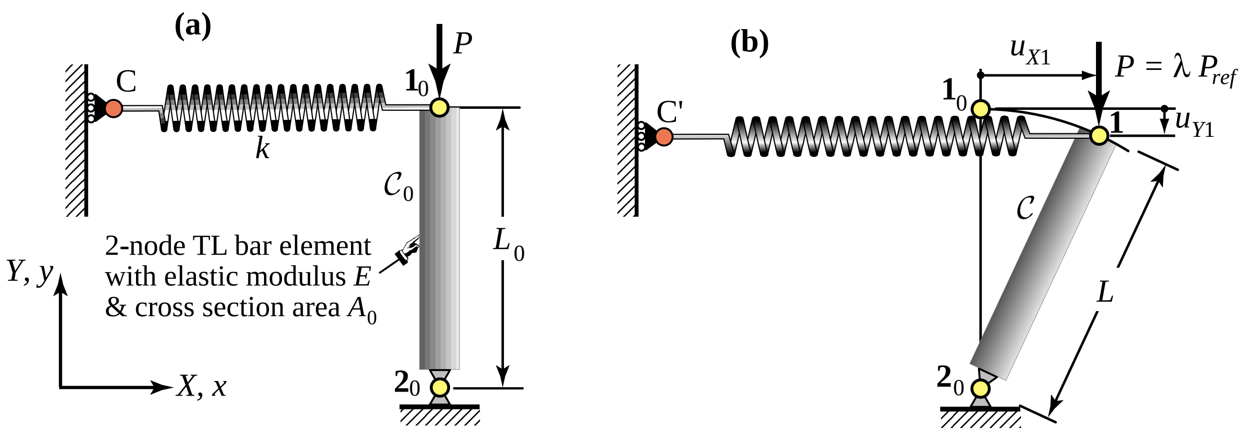
   The eigenvalue $\lambda_i$ closest to zero is the critical load factor, while the associated eigenvector $\mathbf{z}_i$ gives the corresponding buckling mode.

# A Simple Example

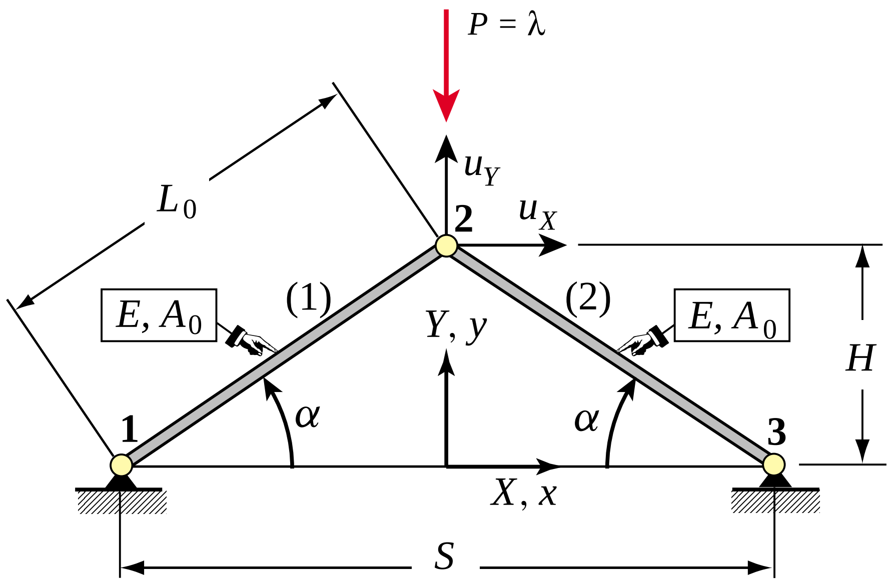



**Can we make the column rigid?   No, the FEM model "blows up".  Why?**

# Other "Deceptively Simple" Problems



# Your Old Friend - the Mises Truss




**LPB predictions for shallow configurations are way wrong. Why?**