28

Structural Stability: Basic Concepts

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§28.1. Introduction

This Chapter presents basic concepts on structural stability, describes how to test it, classifies models and analysis methods, and concludes by contrasting exact versus linearized determination of critical loads using the equilibrium method. Subsequent chapters focus on FEM stability analysis.

The term *stability* has informal and formal meanings. As regards the former, the *American Heritage Dictionary* lists the following three: **1.** Resistance to sudden change, dislodgment, or overthrow; **2a.** Constancy of character or purpose: tenacity, steadfastness; **2b.** Reliability, dependability. Verb: *to stabilize*, Adjective: *stable*. Antonyms: *stability loss, instability, to destabilize, unstable*.f

The formal meaning is found in engineering and sciences. It concerns the stability of *systems*.[1] Structural stability can be informally defined as

The power to recover equilibrium.

It is an essential requirement for all structures. Jennings [422, Chapter 7], provides the following historical sketch:

> "Masonry structures generally become more stable with increasing dead weight. However when iron and steel became available in quantity, elastic buckling due to loss of stability of slender members appeared as a particular hazard."

§28.2. Terminology

Readers only interested in the stability portion of this book, which starts with the present chapter, may want to skim Tables 25.1 and 1.2 in Chapter 1. Those collect often used terms.

In this and following chapters we shall only use physical *kinematic variables*, such as displacements and rotations, as degrees of freedom (DOF). We shall generically denote the DOF variables by $\mathbf{u}$. In discrete models, $\mathbf{u}$ is simply the vector that collects all DOF, and is called the *state vector*. In continuous models, $\mathbf{u}$ is a *function* of the spatial (position) coordinates.

§28.3. Testing Stability

The stability of a mechanical system, and of structures in particular, can be tested (experimentally or analytically) by observing how it reacts when external disturbances are applied. The semi-informal definition given in many textbooks is

A structure is stable at an equilibrium position if it returns to that position upon being disturbed by an extraneous action.

(28.1)

This definition is akin to that given in Table 1.2. It conveys the right flavor, but needs to be made more precise. For that we need to distinguish between static and dynamic equilibrium.

[1] As defined in Table 25.1, a *system* is a functionally related group of components forming or regarded as a collective entity. This definition uses "component" as a generic term that embodies "element" or "part," which connote simplicity, as well as "subsystem," which connotes complexity. In this book we are concerned about *mechanical systems* governed by Newtonian mechanics, with focus on *structures*.

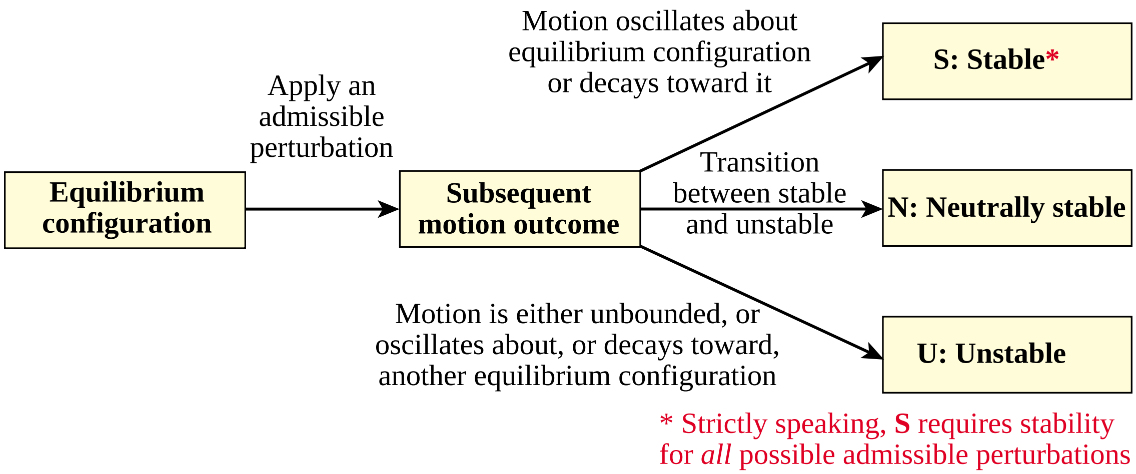

FIGURE 28.1. Outcomes of stability test done on a static equilibrium configuration.

§28.3.1. Stability of Static Equilibrium

For simplicity we will assume that the structure under study is *elastic*, since memory and historical effects such as plasticity or creep introduce complications such as path dependence, which are beyond our scope. The applied forces are characterized by a *load factor* λ, also called a *load parameter* or *load multiplier*. This value scales *reference loads* to provide the actual applied loads.

Setting $\lambda = 0$ means that the structure is unloaded and takes up an equilibrium configuration $\mathcal{C}_0$ called the *undeformed state*. We assume that this state is *stable*. As λ is monotonically varied (either up or down) the structure deforms and assumes equilibrium configurations $\mathcal{C}(\lambda)$. These are assumed to be (i) continuously dependent on λ, and (ii) stable for sufficiently small $|\lambda|$.

How is stability tested? Freeze λ at a value, say λ_d, in which d connotes "deformed." The associated equilibrium configuration is $\mathcal{C}_d = \mathcal{C}(\lambda_d)$. Apply a *perturbation* to $\mathcal{C}_d$, and *remove* it. What sort of perturbation? Any action that may disturb the state, for example a tiny load or a small imposed motion. It must meet two conditions: any kinematic constraints (for example structural supports) must be satisfied, and the applied loads are kept fixed. Such perturbations are qualified as *allowed* or *admissible*, as noted in Table 1.2.[2]

Applying and removing an allowed perturbation will trigger subsequent motion of the system. Three possible outcomes are sketched in Figure 28.1.

S: Stable For *all* admissible perturbations, the structure either returns to the tested configuration $\mathcal{C}_d$ or executes bounded oscillations about it. If so, the equilibrium is called *stable*.

U: Unstable If for at least *one* admissible perturbation the structure moves to (decays to, or oscillates about) another configuration, or "takes off" in an unbounded motion, the equilibrium is *unstable*.

N: Neutral The transition from stable to unstable occurs at a value λ_{cr}, which is called the *critical load factor*. The configuration $\mathcal{C}_{cr} = \mathcal{C}(\lambda_{cr})$ at the critical load factor is

[2] In the framework of variational calculus, those perturbations are known as *admissible variations*, hence the name.

said to be in *neutral* equilibrium. The quantitative determination of this transition is a key objective of the stability analysis.

The foregoing classification has gaps and leaves some details unanswered.

First, speaking about "moving" or "returning" introduces time into the picture. Indeed the concept of stability is necessarily *dynamic* in nature.[3] There is a *before*: the act of applying the perturbation to the frozen configuration, and an *after*: what happens upon removing it. Many practical methods to assess critical loads, however, *factor out the time dimension* as long as certain conditions — notably conservative loading— are verified. Those are known as *static criteria*.

Second, the concept of "perturbation" as "small imposed change" is imprecise. How small is a "tiny load" or a "slight deflection"? The idea will be made more mathematically precise later when we introduce *linearized stability*, also called "stability in the small." This is a natural consequence of assuming *infinitesimal* configuration changes.

§28.3.2. Stability of Dynamic Equilibrium

Stability of motion is a more general topic that includes the static case as a particular one. (As previously noted, the concept of stability is essentially dynamic in nature.)

Suppose that a mechanical system is moving in a *predictable* manner. For example, a bridge or tower oscillates under wind, an airplane is flying a predefined trajectory under automatic pilot, a satellite orbits the Earth, the Earth orbits the Sun. What is the sensitivity of such a motion to changes of parameters such as initial conditions or force amplitude? If the system includes stochastic or chaotic elements, like turbulence, the analysis may also require probabilistic methods.

To make such problems mathematically tractable one usually restricts the kind of admissible motions in such a way that a *bounded reference motion* can be readily defined. For example, harmonic motions of a structure oscillating under periodic excitation. Departures under parametric changes are studied. Transition to unbounded or unpredictable motion is taken as a sign of instability.

An important application of this concept are vibrations of structures that interact with external or internal fluid flows: bridges, buildings, airplanes, antennas, fluid pipes. The steady speed of the flow may be taken as parameter. At a certain flow speed, increasing oscillations may be triggered: this is called *flutter*. Or a non-oscillatory unbounded motion occurs: this is called *divergence*. A famous example of flutter in a civil structure was the collapse of the newly opened Tacoma-Narrows suspension bridge near Seattle in 1940 under a moderate wind speed of about 40 mph.

Modeling and analysis of dynamic instability is covered the last four chapters. In this and next 5 chapters, attention is focused on *static* instability.

[3] For example, [65, p. 144] states "Failure of structures is a dynamical process, and so it is obviouly more realistic to approach buckling and instability from a dynamical point of view."

External Loading	Critical State (math name)	Instability Type (engineering name)	Mathematical Characterization of Instability Event
Conservative (e.g. gravity)	**Bifurcation**	**Buckling**	**Two or more equilibrium branches intersect**
	Limit point	**Snap (aka snap-through)**	**Equilibrium branch reaches maxmin or inflexion w.r.t. control parameter**
Nonconservative (e.g. friction)	**Vanishing frequency**	**Divergence**	**A frequency pair passes through zero & one root becomes positive real***
	Frequency coalescence	**Flutter**	**Two frequency pairs coalesce & one root gets a positive real part***

***** In the frequency complex plane. The conservative case does not require complex arithmetic.

FIGURE 28.2. Four scenarios in which structural stability may be lost. It includes only those treated in this book. It excludes scenarios not covered, such as nonlinear material behavior (plasticity, creep, fracture), parametric excitation by periodically varying loads, and trajectory stability.

§28.4. Static Stability Loss

On restricting attention to *static stability*, two behavioral assumptions are introduced:

Linear Elasticity. The structural material is, and remains, linearly *elastic*. Displacements and rotations, however, are not necessarily small.

Conservative Loading. The applied loads are *conservative*, that is, derivable from a potential. For example, gravity and hydrostatic loads are conservative. On the other hand, aerodynamic and propulsion loads (wind gusts on a bridge, rocket thrust, etc) are generally nonconservative.

The chief reason for the second restriction is that loss of stability under nonconservative loads is inherently *dynamic in nature*: time cannot generally be factored out. As a consequence, it requires additional information, such as mass properties, as well as more advanced mathematics, including the use of complex arithmetic.

§28.4.1. Buckling or Snapping?

Under the foregoing restrictions, two types of instability may occur:

Bifurcation. Structural engineers prefer the more familiar name *buckling* for this one. The structure reaches a *bifurcation point*, at which two or more equilibrium paths intersect. What happens after the bifurcation point is reached is called *post-buckling* behavior.

Snapping. Structural engineers use the closely related term *snap-through* or *snap buckling* for this one. The structure reaches a *limit point* at which the load, or the loading parameter, reaches a maximum or minimum. What happens after that is called *post-snapping* behavior.

Bifurcation points and *limit points* are instances of *critical points*. The importance of critical points in static stability analysis stems from the following property:

$$\boxed{\text{Transition from stability to instability can only occur at critical points}}$$

Reaching a critical point may lead to immediate destruction (collapse) of the structure. This depends on its post-buckling or post-snapping behavior, and on material resilience (brittle or ductile). For some scenarios this information is important as immediate collapse may lead to loss of life.

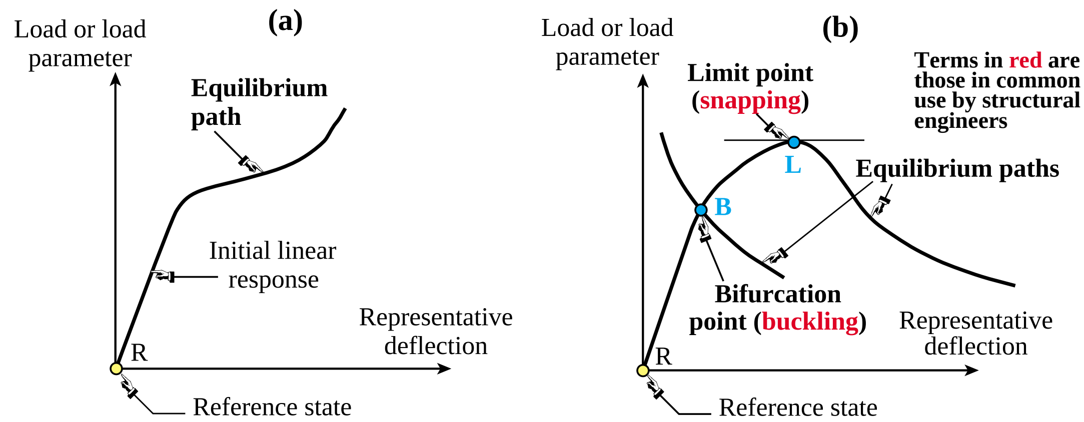

FIGURE 28.3. Graphical representation of static equilibrium paths and critical points: (a) a response path with no critical points; (b) multiple response paths showing occurrence of two types: bifurcation point (B) and limit point (L).

On the other hand, there are some configurations where the structure keeps resisting the critical load, or even taking up increasing loads, after traversing a bifurcation point.[4] Such "load-sustaining" designs are obviously preferable from a safety standpoint.

Figure 28.2 lists four scenarios treated in this book, in which stability of equilibrium can be lost. Note that identifiers used in applied mathematics and structural engineering are generally distinct.

§28.4.2. Response Diagrams

To illustrate the occurrence of static instability as well as critical points we will often display load-deflection response diagrams. These were introduced before (in Chapter 2) but for self-containedness will be reviewed again here. A response diagram is a plot of *equilibrium configurations* taken by a structure as a load, or load parameter, is continuously varied. That is plotted along the vertical axis while a judiciously chosen representative deflection, which could also be an angle of rotation, is plotted along the horizontal axis. A common convention is to take zero deflection at zero load. This defines the *reference state*, labeled as point R in such plots.

A continuous set of equilibrium configurations forms an *equilibrium path*. Such paths are illustrated in Figure 28.3. The plot in Figure 28.3(a) shows a response path with no critical points. On the other hand, that in Figure 28.3(b) depicts the occurrence of two critical points: one bifurcation and one limit point. Those points are labeled as B and L, respectively, in the Figure.

§28.4.3. Primary Equilibrium Path and the Design Critical Load

A complex structure in general will exhibit several critical points, with a mixture of bifurcation and limit points. An important question for design engineers is:

Which critical point should be chosen to establish safety factor against instability?

Most textbooks say: pick the one associated with the lowest[5] critical load or load factor. That is fine if post-buckling behavior is not considered. For a more comprehensive answer, it is useful to introduce the following definition:

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[4] Post-snapping stability is not possible upon crossing a limit point of maxmin type.

[5] If the load factor can take either sign, as happens in some scenarios (for example, shear buckling of panels), *lowest* is taken in the sense of absolute value, i.e., that closest to zero.

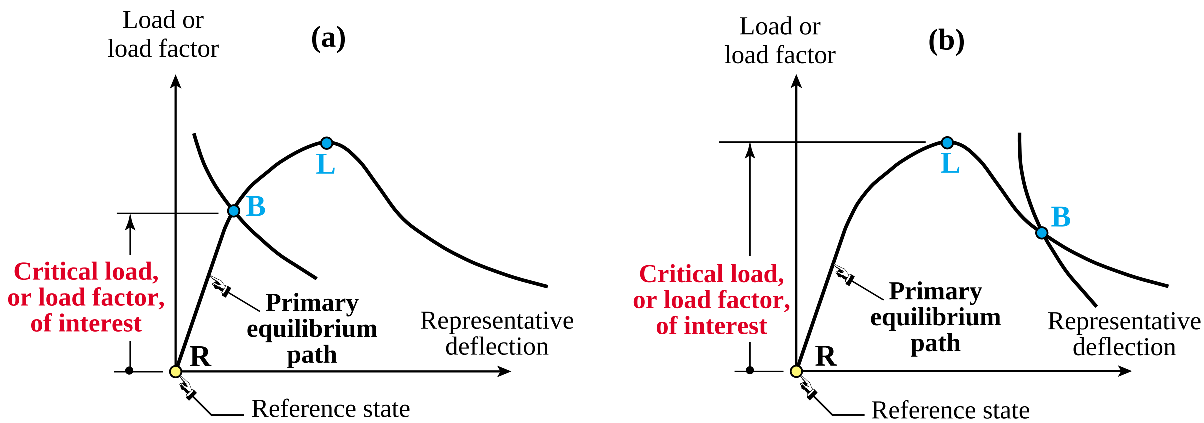

FIGURE 28.4. For design purposes, a critical load is that associated with the critical point first encountered when traversing the primary equilibium path from the reference state. In (a) the critical load occurs at bifurcation point B beacuse the limit point occurs later. In (b) the critical load occurs at the limit point L, because bifurcation occurs later.

The *primary equilibrium path* is the one that passes through the reference state. This is often the same as the undeformed or unloaded state.

We accordingly define the *design critical load* as follows: that *located on the primary equilibrium path that is nearest to the reference state*. This makes engineering sense since most structures are designed to operate on the primary equilibrium path while in service. See Figure 28.4. That location is called the *first critical point*, often abbreviated to FCP. See Table 1.2 for related nomenclature.

Note that in the case of Figure 28.4(b) the limit point L defines the design critical load because it is encountered first while traversing the primary equilibrium path starting from R. It does not matter that bifurcation point B occurs at a lower load factor unless post-critical behavior is important in design — which is rarely the case.

§28.4.4. Stability Models

Stability models of actual structures fall into two categories:

Continuous. Such models have an *infinite* number of degrees of freedom (DOF). They lead to ordinary or partial differential equations (ODEs or PDEs) in space, from which stability equations may be derived by perturbation techniques. Obtaining nontrivial solutions of the perturbed equations generally leads to trascendental eigenproblems, even if the underlying model is linear.

Discrete. These models have a *finite* number of DOF in space. Where do these come from? They usually emerge as discrete approximations to the underlying continuum models. Two common discretization techniques are:

(1) *Lumped parameter models*, in which the flexibility of the structure is localized at a finite number of places. A common model of this type for columns: joint-hinged rigid struts supported by extensional or torsional springs at the joints. See examples in §28.5 and §28.6

(2) *Finite element models* that include the so-called *geometric stiffness* effects. These are covered in the following chapters.

Stability equations for discrete models may be constructed using various devices. For lumped parameter models one may resort to either perturbed equilibrium equations built via FBDs, or to energy methods.[6] For FEM models only energy methods are practical. All techniques eventually lead to matrix stability equations that take the form of an algebraic eigenproblem.

§28.4.5. Stability Equations Derivation

The equations that determine critical points are called *characteristic equations* in the applied mathematics literature.[7] In structural engineering the names *stability equations* and *buckling equations* are common. Two methods are favored for deriving those equations.

Equilibrium Method. The equilibrium equations of the structure are established in a *perturbed configuration*. This is usually done with Free Body Diagrams (FBD). The resulting equations are examined for the occurrence of *nontrivial solutions*. Those are obtained by disturbing the original equilibrium positions through admissible perturbations. If those equilibrium equations are linearized for small perturbations, one obtains an algebraic eigenproblem. The eigenvalues give values of critical loads while eigenvectors yield the buckling or snapping mode shapes.

Energy Method. The total potential energy of the system is established in terms of its degrees of freedom (DOF). The Hessian matrix of the potential energy function, taken with respect to those DOF, is established and tested for positive definiteness as the load parameter (or set of parameters) is varied. Details are given in §29.3.1. Loss of such property occurs at critical points. These may be in turn categorized into bifurcation and limit points according to a subsequent eigenanalysis.

The energy method is more general for structures subject to conservative loading. It has two major practical advantages:

(1) Merges naturally with the FEM formulation because the Hessian of the potential energy is the tangent stiffness matrix. Thus it can be readily implemented in general-purpose FEM codes.

(2) Requires no *a priori* assumptions as regards admissible perturbations.

One clear advantage of the equilibrium method is physical transparency. That makes it invaluable for undergraduate teaching. The energy method occults the physics behind eigensmoke clouds.

§28.5. Single-DOF Equilibrium Analysis Examples

This section illustrates the *geometrically exact equilibrium method* of stability analysis with three examples that involve single-DOF lumped-parameter models. The key feature is that Free Body Diagrams (FBD) must take the exact post-buckling configuration into account. The results are compared with those given by the Linearized Prebuckling (LPB) simplified analysis.

Since examples involve rigid members, a conventional FEM-DSM discretization cannot be directly used, as it would involve infinite stiffness entries. See Chapter 29 for a FEM treatment.

[6] If the lumped parameter model contains rigid members, as is often the case, the DSM version of FEM is not easily applicable because stiffness matrices of rigid elements would have infinite entries. This shortcoming may be addressed using either multifreedom constraints, or FEM formulations equivalent to the equilibrium method. But the latter choice rules out the use of commercial DSM-based codes.

[7] Often this term is restricted to the determinantal form of the stability eigenproblem. This is the equation whose roots give the eigenvalues that can be interpreted as critical loads or critical load parameters.

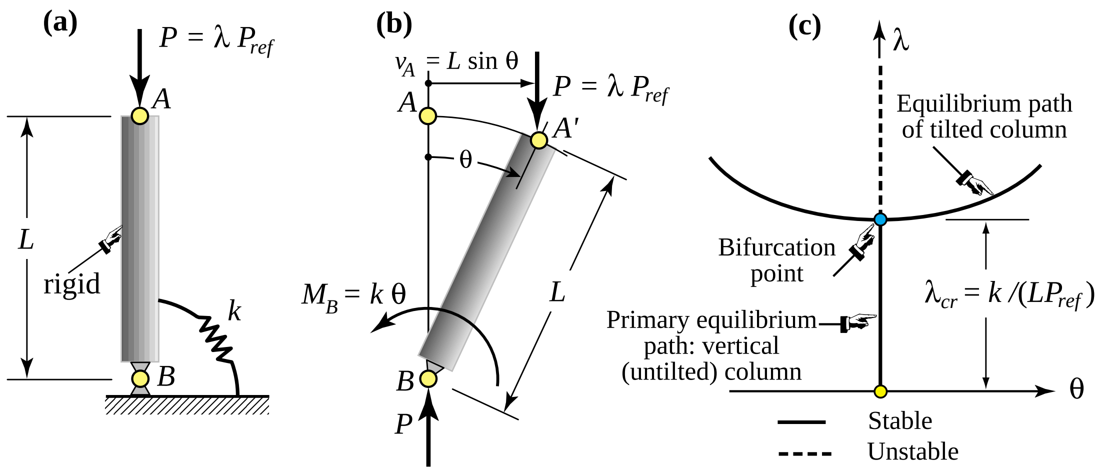

FIGURE 28.5. Geometrically exact stability analysis of a torsional-spring-propped hinged cantilevered rigid (TSPHRC) column: (a) untilted (reference) configuration, (b) FBD of tilted column, (c) equilibrium paths (full line: stable, dashed line: unstable).

§28.5.1. TSPHRC Column: Geometrically Exact Analysis

Consider the configuration depicted in Figure 28.5(a). A *rigid* column of length L is stabilized (propped) by a torsional spring of stiffness $k > 0$. It is axially loaded by a vertical dead load $P = \lambda P_{ref}$, in which P_{ref} is a reference load and λ a dimensionless load parameter. The load remains *vertical* as the column tilts. (Note that k has the physical dimension of force $\times$ length.) This configuration will be called a *torsional-spring-propped hinged rigid cantilever* column, or TSPHRC column for brevity. The definition $P = \lambda k/L$ renders λ dimensionless, which is convenient for result presentation. The tilt angle θ, positive CW, is picked as state parameter (and only DOF) as most appropriate for the ensuing analysis.

For sufficiently small P the column remains vertical as in Figure 28.5(a), with $\theta = 0$. The only possible buckled shape is the tilted column shown in Figure 28.5(b). That figure depicts the FBD required to work out the equilibrium of the tilted configuration. Notice that θ is *not* assumed small. Taking moments with respect to the hinge B we obtain the following equilibrium equation in terms of λ and θ:

$$k\,\theta = P\,v_A = \lambda\,P_{ref} L\,\sin\theta \quad\Rightarrow\quad k\,\theta - \lambda\,P_{ref}\,L\,\sin\theta = 0. \tag{28.2}$$

The equation on the right has two equilibrium solutions:

$$\boxed{\theta = 0 \;\text{ for any } \lambda, \qquad \lambda = \frac{k}{P_{ref}\,L}\,\frac{\theta}{\sin\theta}.} \tag{28.3}$$

These pertain to the untilted ($\theta = 0$) and tilted ($\theta \neq 0$) equilibrium paths, respectively. Since $\lim(\theta/\sin\theta) \to 1$ as $\theta \to 0$, the paths intersect when

$$\boxed{\lambda_{cr} = \frac{k}{P_{ref}\,L}, \quad \text{or} \quad P_{cr} = \frac{k}{L}.} \tag{28.4}$$

The two paths are plotted in Figure 28.5(c). The intersection (28.4) characterizes a *bifurcation point* B. Four equilibrium branches emanate from B. Three are stable (full line) and one is unstable (dashed line). Observe that the applied load may *rise* beyond $P_{cr} = \lambda_{cr}\, P_{ref} = k/L$ by moving to a tilted configuration. It is not difficult to show that the maximum load occurs if $\theta \rightarrow 180^\circ$, for which $P \rightarrow \infty$; this is a consequence of the assumption that the column is rigid, and that it may fully rotate by that amount without being impeded, say, by hitting the ground.

§28.5.2. TSPHRC Column: LPB Analysis

The geometrically exact analysis that leads to (28.3) has the advantage of providing a complete solution. In particular, it shows what happens after the bifurcation point B is traversed; a behavior called *post buckling*. For this configuration the structure maintains load-bearing capabilities while tilted, which is the hallmark of a safe design. But for more complicated problem this approach becomes impractical as it involves solving systems of nonlinear algebraic or differential equations. Post-buckling analysis has then to rely on geometrically nonlinear FEM.

Often the engineer is interested only in the critical load. This is especially true in preliminary design scenarios, when a key objective is to assess safety factors against buckling. If so, it is more practical to work with a *linearized* version of the problem. The technical name is *Linearized Prebuckling* (LPB) analysis. This approach relies on the following assumptions.[8]

- Deformations prior to buckling are neglected. Consequently the analysis can be carried out in the *reference configuration* (undeformed) geometry.
- Perturbations of the reference configuration are restricted to *infinitesimal* displacements and rotations.
- The structure remains linearly elastic up to buckling.
- Both structure and loading do not exhibit any imperfections.
- The critical state is a bifurcation point.

We apply these rules to the TSPHRC column of Figure 28.5(a). The equilibrium equation (28.2) is linearized by assuming an infinitesimal tilt angle $\theta << 1$, whence $\sin\theta \approx \theta$ and that expression becomes

$$\big(k - \lambda\, P_{ref}\, L\big)\, \theta = 0. \tag{28.5}$$

This is the *LPB stability equation*. Since the product of two numbers is zero, at least one must be zero: $\theta = 0$, or $k - \lambda\, P_{ref}/L = 0$, or both. The solution $\theta = 0$ reproduces the untilted configuration. For buckling to occur, we must have $\theta \neq 0$. If so, the expression in parenthesis must vanish, which requires

$$\boxed{\lambda = \lambda_{cr} = \frac{k}{P_{ref}\, L} \quad \text{or} \quad P_{cr} = \lambda_{cr}\, P_{ref} = \frac{k}{L}.} \tag{28.6}$$

This reproduces the critical load given in (28.4).

Note that a LPB analysis only provides the critical load. It *does not give any information on post-buckling behavior*. If this is necessary, a more comprehensive analysis, such as the geometrically exact one carried out in the previous subsection, is required.

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[8] These assumptions are discussed critically in the following chapters.

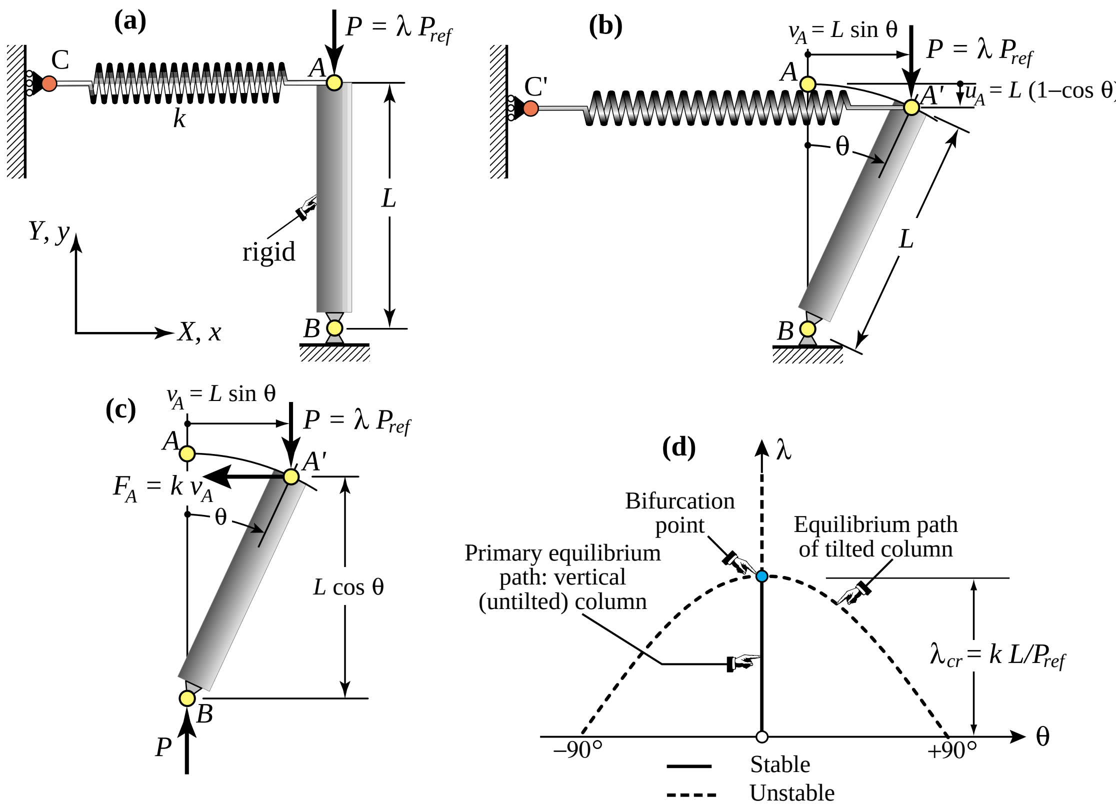

FIGURE 28.6. Geometrically exact analysis of a extensional-spring-propped hinged cantilever rigid (ESPHRC) column with extensional spring remaining horizontal: (a) untilted column, (b) tilted column showing spring remaining horizontal, (c) FBD of tilted column, (c) equilibrium paths (full line: stable, dashed line: unstable).

It should also be noted that *if the loss of stability is by snap buckling, it cannot be obtained by LPB analysis*. The underlying reason is that finite deformations prior to buckling are essential in finding limit points, whereas the first LPB assumption listed above explicitly precludes those.

§28.5.3. ESPHRC Column: Geometrically Exact Analysis

Consider next the configuration pictured in Figure 28.6(a). This one differs from the TSPHRC column in the type of stabilizing spring. A rigid strut of length L is hinged at B and supports a vertical load $P = \lambda P_{ref}$ at end A. The load remains *vertical* as the column tilts. The column is propped by an extensional spring of stiffness k attached to A. This configuration will be called a *extensional-spring-propped hinged rigid cantilevered* column; or ESPHRC column for short. As before, the only DOF is the tilt angle θ.

For the geometrically exact analysis is it important to know what happens to the spring as the column tilts. One possible assumption is that it remains horizontal, as pictured in Figure 28.6(b). If so, the FBD in the tilted configuration will be as shown in n Figure 28.6(c). Taking moments about B yields the equilibrium condition

$$\lambda P_{ref}\ \sin\theta = k\,L\ \sin\theta\ \cos\theta. \tag{28.7}$$

(Important: *do not* cancel out $\sin\theta$ from both sides of (28.7) yet — that will cause the primary equilibrium path $\theta = 0$ to be lost as a solution.) This equation has two equilibrium solutions

$$\boxed{\theta = 0 \;\text{ for any } \lambda\,, \quad \lambda = \frac{k\,L}{P_{ref}}\,\cos\theta.} \tag{28.8}$$

These solutions yield the vertical-column (primary) and tilted-column (secondary) equilibrium paths, which are plotted in Figure 28.6(d) on the λ versus θ plane. The two paths intersect at $\theta = 0$ and $\lambda = \lambda_{cr} = k\,L/P_{ref}$, which is a bifurcation point. Consequently the critical load is given by

$$\boxed{\lambda_{cr} = \frac{k\,L}{P_{ref}} \quad \text{or} \quad P_{cr} = \lambda_{cr}\,P_{ref} = k\,L.} \tag{28.9}$$

Of the four branches that emanate from B, only one (the primary $\theta = 0$ path for $\lambda < \lambda_{cr}$) is stable. Once B is reached, the tilted column supports only a decreasing load P, which vanishes at $\theta = \pm 90^\circ$. Consequently this configuration is poor from the standpoint of post-buckling safety.

Another reasonable assumption is that the spring attachment point to the wall, which is called C in Figure 28.7(a), stays fixed. The distance AC is parametrized with respect to the column length as ηL, in which η is dimensionless. If the column tilts, the spring also tilts as pictured in Figure 28.7(b).

The geometrically exact FBD for this case is shown in Figure 28.7(c). As can be observed, it is considerably more involved than for the spring-stays-horizontal case. We quote only the final result for the two equilibrium path equations:

$$\boxed{\theta = 0 \quad \text{for any } \lambda, \quad \lambda = \frac{kL}{P_{ref}}\,\frac{\eta\cos\theta + \sin\theta}{\eta + \sin\theta}.} \tag{28.10}$$

As in previous cases, the first solution corresponds to the untilted column whereas the second one pertains to the tilted one. Figure 28.8 shows response plots on the λ versus θ plane for the tilted column, that is, the second solution in (28.10) for the four cases $\eta = \frac{1}{4}$, $\eta = \frac{1}{2}$ $\eta = 1$ and $\eta = 5$, drawn with $P_{ref} = 1$, $k = 1$, and $L = 1$.

Although the bifurcation points stay in the same location: $\lambda = 1$ and $\theta = 0$, the post-buckling response is no longer symmetric with respect to θ. That deviation is most conspicuous when $\eta < 1$, since if so the spring tilting has a highly noticeable effect if $\theta < 0$. The sharp drop of λ towards $-\infty$ occurs when the spring and the tilted column are nearly aligned, because if that happens the column is no longer propped. Comparing the $\eta = 5$ plot with that in Figure 28.6(d), it is clear that as $\eta >> 1$ the response approaches that of the spring-stays-horizontal column in (28.8). This may be mathematically proven by taking the limit of the second of (28.10) as $\eta \to \infty$.

§28.5.4. ESPHRC Column: LPB Analysis

To linearize the ESPHRC problem, assume again that θ is so small that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Then the equilibrium equations of the two foregoing cases (horizontal spring and wall-attached spring) collapse to

$$(\lambda\,P_{ref} - k\,L)\,\theta = 0. \tag{28.11}$$

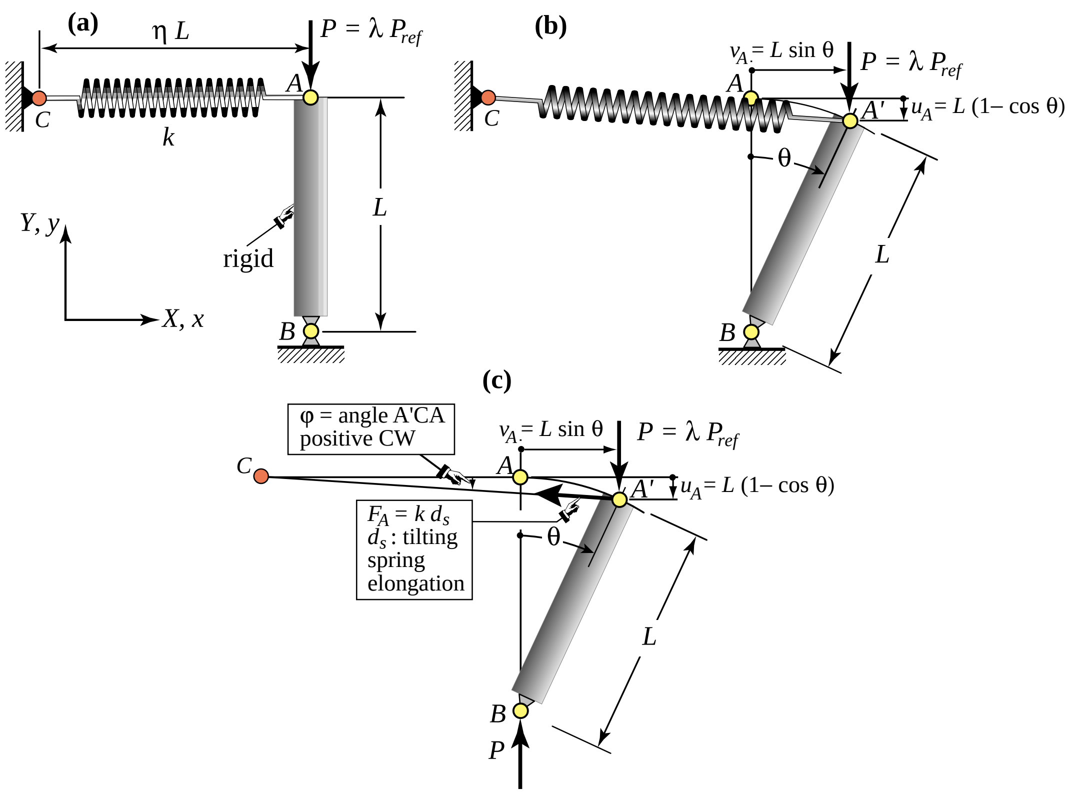

FIGURE 28.7. Geometrically exact stability analysis of a extensional-spring-propped hinged rigid cantilever (ESPHRC) column with wall-attached extensional spring: (a) untilted column; (b) tilted column; (c) FBD for tilted equilibrium.

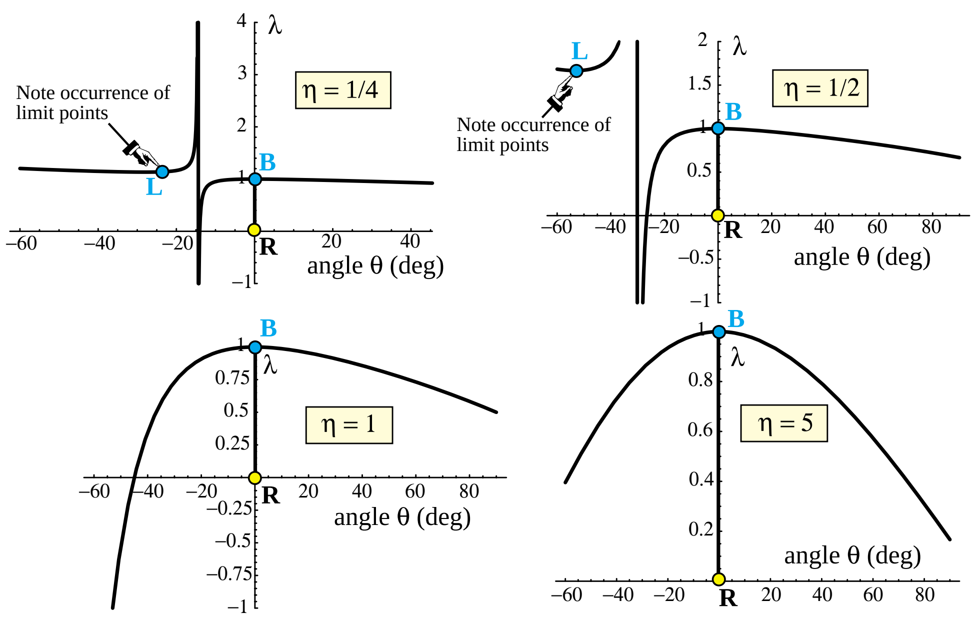

FIGURE 28.8. Geometrically exact analysis of a ESPHRC column with wall-attached extensional spring: λ vs./ θ response diagrams for tilted column and four values of η [defined in Figure 28.7(a)], with $P_{ref} = k = L = 1$. Note: the untilted-column equilibrium path $\theta = 0$ is not plotted beyond B to reduce clutter.

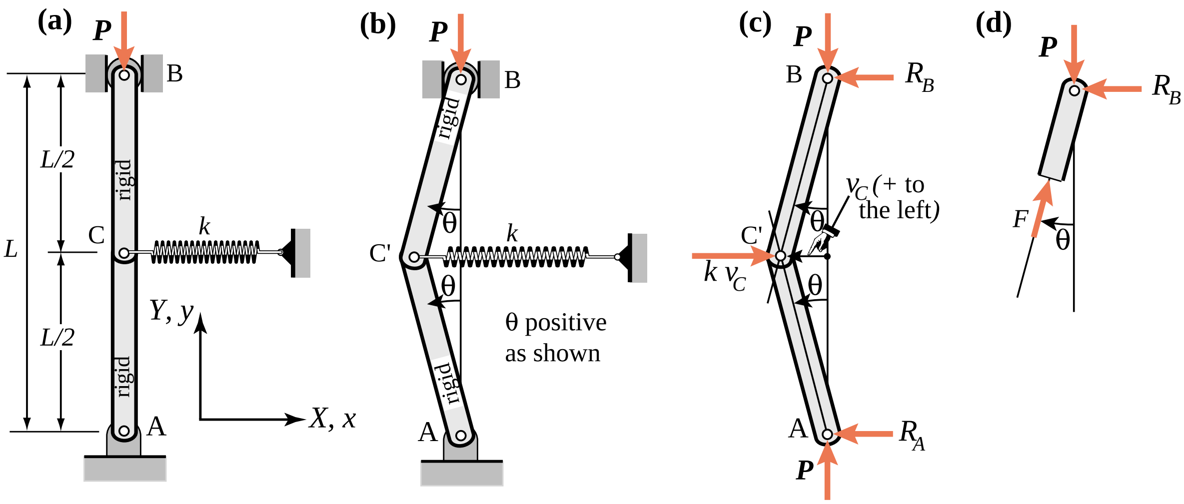

FIGURE 28.9. Geometrically exact stability analysis of two-rigid-strut pinned-pinned column propped by a wall-attached extensional spring: (a) untilted column; (b) tilted (buckled) column; (c) FBD for tilted equilibrium of whole column; (d) FBD of strut BC.

The two solutions of (28.11) represent the equilibrium paths $\theta = 0$ and $\lambda = k\,L/P_{ref}$, which pertain to the untilted (vertical) and tilted column, respectively. Consequently

$$\boxed{\lambda_{cr} = \frac{k\,L}{P_{ref}}, \qquad \text{or} \qquad P_{cr} = \lambda_{cr}\,P_{ref} = k\,L.} \tag{28.12}$$

This result is *independent* of assumptions on how the spring wall-attachment point behaves after the column buckles. This is to be expected since linearization filters out that information. Once again, the LPB analysis provides no information on post-buckling behavior.

§28.5.5. Two-Strut Propped Column: Geometrically Exact Analysis

The column shown in Figure 28.9(a) is fabricated with two equal-length rigid struts and propped by a lateral extensional spring of stiffness k at its mid-hinge. It is pinned at both ends and compressed by load P at the top. Find the critical load by geometrically exact bifurcation analysis.

The tilt angle θ defined in Figure 28.9(c), positive as shown, is taken as the only degree of freedom (DOF). The buckling shape can only be as sketched in Figure 28.9(b), except that it may go either way. Assuming θ is finite for an exact analysis, the spring elongates by $v_C = (L/2)\sin\theta$. The restoring spring force is $kv_C = k(L/2)\sin\theta$, which acts opposite to the assumed lateral deflection, as indicated in the whole-column FBD of Figure 28.9(c). From the FBD of link-BC shown in Figure 28.9(d), horizontal force equilibrium gives $F\cos\theta = P$ and $R_B = F\sin\theta = P\tan\theta$. Taking moments with respect to bottom hinge A gives the stability equation:

$$\begin{aligned}\sum M_A &= R_B\left(\frac{L}{2}\cos\theta + \frac{L}{2}\cos\theta\right) - \frac{kL}{2}\sin\theta\,\frac{L}{2}\cos\theta = P\tan\theta\,L\cos\theta - \frac{kL^2}{4}\sin\theta\cos\theta \\ &= L\sin\theta\left(P - \frac{kL}{4}\cos\theta\right) = 0.\end{aligned} \tag{28.13}$$

This exact equilibrium equation has two solutions: $\theta = 0$ for any P, in which the column stays vertical, and $P = \frac{1}{4} k\,L\, \cos\theta$, which pertains to the tilted equilibrium configuration. The two equilibrium paths intersect in the $\{P, \theta\}$ plane at the bifurcation point

$$\boxed{\theta = 0, \quad P = P_{cr} = \tfrac{1}{4} k\,L.} \tag{28.14}$$

Since $\cos\theta$ decreases as θ moves away from 0 (either way), the post-buckling behavior is inherently unstable, and the configuration may be classified as unsafe.

The LPB analysis of this problem is posed as an Exercise.

§28.6. Multiple-DOF Equilibrium Analysis Examples

This section presents an example of the equilibrium method applied to lumped parameter models with more than one DOF. Only the LPB analysis is carried out.[9] It will be seen that a *stability matrix* emerges, from which an *eigenproblem* that yields critical loads and mode shapes can be immediately set up.

§28.6.1. PP3S Column

Figure 28.10(a) shows an axially loaded pinned-pinned column of length L built with three rigid struts of equal length. This will be referred to as the PP3S column. The struts are pinned at joints A through D. The column is propped by two extensional springs of stiffness k working as elastic supports at joints B and C. Determine the critical loads by Linearized PreBuckling (LPB) analysis.

Figure 28.10(b) depicts how the structure might realistically displace upon buckling. Because the struts are rigid and cannot change length, joints A, B and C will necessarily move downward while the springs tilt. Such a configuration would be correct for a *geometrically exact* analysis.

The LPB simplification assumes infinitesimal displacements from the reference state, as shown in Figure 28.10(c). Here B and C move horizontally by v_B and v_D, respectively, whereas A does not move at all. The horizontal reactions R_A and R_D shown there are obtained in terms of F_B and F_D by statics on taking moments with respect to D and A, respectively, in the deformed configuration.

The two FBDs used to form the stability equations are shown in Figure 28.11. We take moments with respect to C' in the left FBD and with respect to B' in the right FBD:

$$\sum M_{C'} = -P\, v_C - F_B \frac{L}{3} + R_A \frac{2L}{3} = 0, \quad \sum M_{B'} = -P\, v_B - F_C \frac{L}{3} + R_D \frac{2L}{3} = 0. \tag{28.15}$$

Substituting $R_A = (2F_B + F_C)/3$, $R_D = (F_B + 2F_C)/3$, $F_B = k\, v_B$ and $F_C = k\, v_C$, and recasting (28.15) in matrix form we obtain the stability equations

$$\frac{k\,L}{9} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_B \\ v_C \end{bmatrix} = P \begin{bmatrix} v_B \\ v_c \end{bmatrix}, \tag{28.16}$$

This is a 2×2 matrix eigenproblem with P as the eigenvariable. The characteristic equation is

$$C(P) = \det\left(\frac{k\,L}{9} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{k^2\,L^2}{27} - \frac{4\,k\,P\,L}{9} + P^2. \tag{28.17}$$

[9] The geometrically exact analysis is too elaborate.

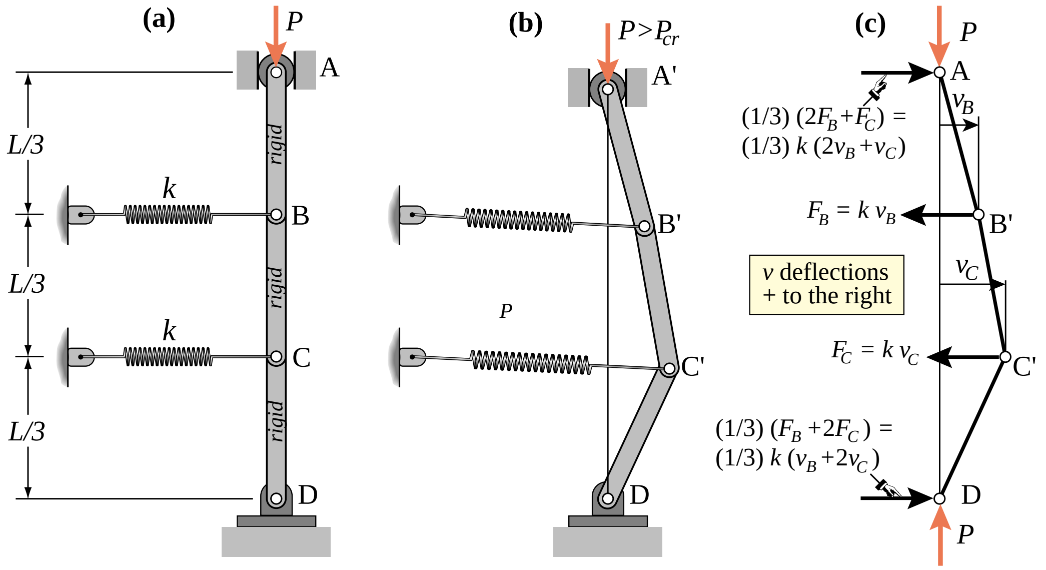

FIGURE 28.10. Stability analysis of pinned-pinned three-strut (PP3S) column propped by extensional springs: (a) reference configuration; (b) admissible buckled shape showing realistic geometry change (struts do not change length); (c) FBD in *linearized* configuration: B and C displace by small *horizontal* motions whereas A stays fixed.

This is a quadratic polynomial in P. The two roots of $C(P) = 0$ give the critical loads:

$$\boxed{P_{cr1} = \frac{kL}{9}, \qquad P_{cr2} = \frac{kL}{3}.} \tag{28.18}$$

An alternative form is obtained on premultiplying (28.16) by the inverse of the LHS matrix in (28.16). This gives

$$\frac{3P}{k\,L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_B \\ v_c \end{bmatrix} = \begin{bmatrix} v_B \\ v_C \end{bmatrix}, \tag{28.19}$$

a standard eigenproblem which gives the same critical loads (28.18).

The eigenvectors corresponding to the critical loads (28.18), normalized to 1 as largest entry, are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{28.20}$$

These are plotted in Figure 28.12. Note that the mode shape corresponding to the lowest critical load, $P_{cr1} = kL/9$, is *antisymmetric*.

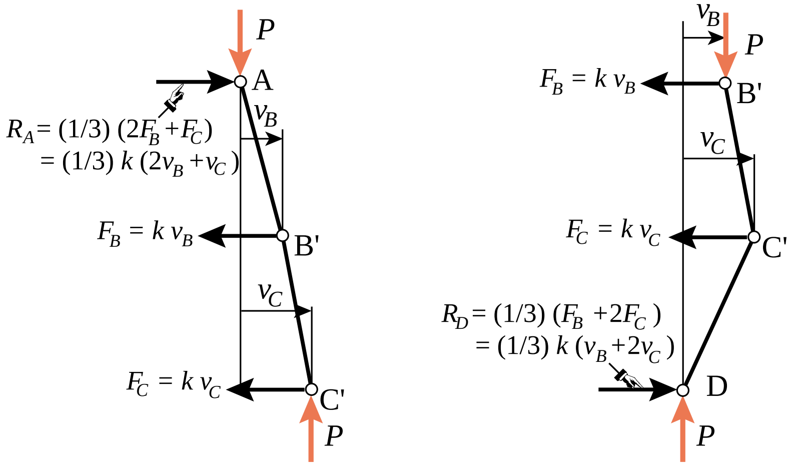

FIGURE 28.11. Free Body Diagrams for the 3-strut column of Figure 28.10.

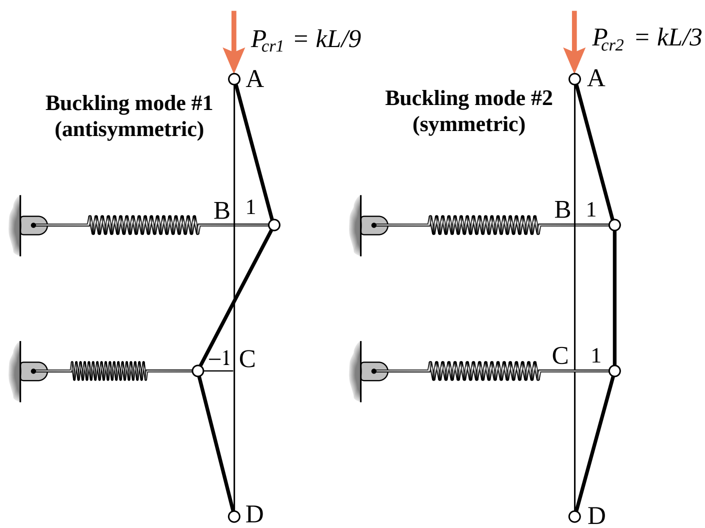

FIGURE 28.12. Critical loads and buckling mode shapes for the 3-strut column of Figure 28.10.

Notes and Bibliography

There is a very large number of books devoted to structural stability. Only two of the better known are cited: Timoshenko and Gere [775] (1961, reprinted by Dover in 2009), and Bazant and Cedolin [65]. The latter is more comprehensive, covering FEM models, inelastic effects and fracture, and has an updated bibliography.

The equilibrium method is preferably to FEM in two contexts: undergraduate teaching level, and lumped-parameter models (rigid struts stabilized by extensional or torsional springs). For these simple models it displays the physics well, as a contest between the bad guys (overturning forces) versus the good guys (resisting springs). For more realistic continuum models, FEM wins hands down, as will be illustrated in subsequent Chapters.