EFFECTIVE MODAL MASS & MODAL PARTICIPATION FACTORS
Revision F

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Introduction

The effective modal mass provides a method for judging the "significance" of a vibration mode.

Modes with relatively high effective masses can be readily excited by base excitation. On the other hand, modes with low effective masses cannot be readily excited in this manner.

Consider a modal transient or frequency response function analysis via the finite element method. Also consider that the system is a multi-degree-of-freedom system. For brevity, only a limited number of modes should be included in the analysis.

How many modes should be included in the analysis? Perhaps the number should be enough so that the total effective modal mass of the model is at least 90% of the actual mass.

Definitions

The equation definitions in this section are taken from Reference 1.

Consider a discrete dynamic system governed by the following equation

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{1}$$

where

M is the mass matrix

K is the stiffness matrix

$\ddot{\bar{x}}$ is the acceleration vector

$\bar{x}$ is the displacement vector

$\bar{F}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let ϕ be the eigenvector matrix.

The system's generalized mass matrix $\hat{m}$ is given by

$$\hat{m} = \phi^{T} M \phi \tag{2}$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^{T} M \, \bar{r} \tag{3}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{4}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\,2}}{\hat{m}_{ii}} \tag{5}$$

Note that $\hat{m}_{ii} = 1$ for each index if the eigenvectors have been normalized with respect to the mass matrix.

Furthermore, the off-diagonal modal mass ($\hat{m}_{ij}, i \neq j$) terms are zero regardless of the normalization and even if the physical mass matrix M has distributed mass. This is due to the orthogonality of the eigenvectors. The off-diagonal modal mass terms do not appear in equation (5), however.

Example

Consider the two-degree-of-freedom system shown in Figure 1, with the parameters shown in Table 1.

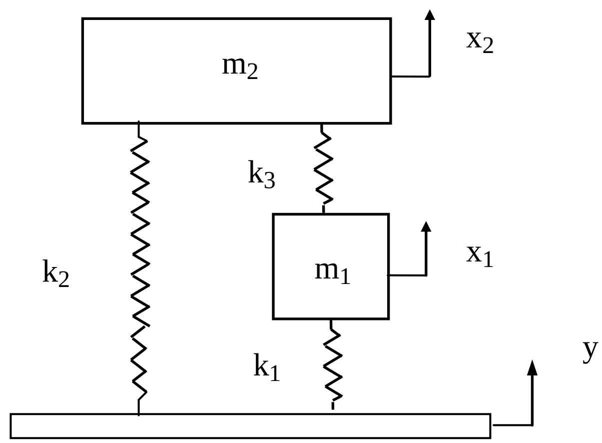

Figure 1.

Table 1. Parameters	
Variable	Value
m_1	2.0 kg
m_2	1.0 kg
k_1	1000 N/m
k_2	2000 N/m
k_3	3000 N/m

The homogeneous equation of motion is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

The mass matrix is

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{kg} \tag{7}$$

The stiffness matrix is

$$K = \begin{bmatrix} 4000 & -3000 \\ -3000 & 5000 \end{bmatrix} N/m \tag{8}$$

The eigenvalues and eigenvectors can be found using the method in Reference 2.

The eigenvalues are the roots of the following equation.

$$\det \left[K - \omega^2 M \right] = 0 \tag{9}$$

The eigenvalues are

$$\omega_1{}^2 = [901.9 \text{ rad/sec}]^2 \tag{10}$$

$$\omega_1 = 30.03 \text{ rad/sec} \tag{11}$$

$$f_1 = 4.78 \text{ Hz} \tag{12}$$

$$\omega_2{}^2 = [6098 \text{ rad/sec}]^2 \tag{13}$$

$$\omega_2 = 78.09 \text{ rad/sec} \tag{14}$$

$$f_2 = 12.4 \text{ Hz} \tag{15}$$

The eigenvector matrix is

$$\phi = \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{16}$$

The eigenvectors were previously normalized so that the generalized mass is the identity matrix.

$$\hat{m} = \phi^T M \phi \tag{17}$$

$$\hat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.6280 & -0.3251 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{18}$$

$$\hat{m} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 1.2560 & -0.6502 \\ 0.4597 & 0.8881 \end{bmatrix} \tag{19}$$

$$\hat{m} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

Again, $\bar{r}$ is the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement. For this example, each mass simply has the same static displacement as the ground displacement.

$$\bar{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{21}$$

The coefficient vector $\bar{L}$ is

$$\bar{L} = \phi^T M \, \bar{r} \tag{22}$$

$$\bar{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{23}$$

$$\bar{L} = \begin{bmatrix} 0.6280 & 0.4597 \\ -0.3251 & 0.8881 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{24}$$

$$\bar{L} = \begin{bmatrix} 1.7157 \\ -0.2379 \end{bmatrix} \text{ kg} \tag{25}$$

The modal participation factor Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{26}$$

The modal participation vector is thus

$$\Gamma = \begin{bmatrix} 1.7157 \\ -0.2379 \end{bmatrix} \tag{27}$$

The coefficient vector $\bar{L}$ and the modal participation vector Γ are identical in this example because the generalized mass matrix is the identity matrix.

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\;2}}{\hat{m}_{ii}} \tag{28}$$

For mode 1,

$$m_{eff,1} = \frac{[1.7157 \text{ kg}]^2}{1 \text{ kg}} \tag{29}$$

$$m_{eff,1} = 2.944 \text{ kg} \tag{30}$$

For mode 2,

$$m_{eff,2} = \frac{[-0.2379 \text{ kg}]^2}{1 \text{ kg}} \tag{31}$$

$$m_{eff,2} = 0.056 \text{ kg} \tag{32}$$

Note that

$$m_{eff,1} + m_{eff,2} = 2.944 \text{ kg} + 0.056 \text{ kg} \tag{33}$$

$$m_{eff,1} + m_{eff,2} = 3 \text{ kg} \tag{34}$$

Thus, the sum of the effective masses equals the total system mass.

Also, note that the first mode has a much higher effective mass than the second mode.

Thus, the first mode can be readily excited by base excitation. On the other hand, the second mode is negligible in this sense.

From another viewpoint, the center of gravity of the first mode experiences a significant translation when the first mode is excited.

On the other hand, the center of gravity of the second mode remains nearly stationary when the second mode is excited.

Each degree-of-freedom in the previous example was a translation in the X-axis. This characteristic simplified the effective modal mass calculation.

In general, a system will have at least one translation degree-of-freedom in each of three orthogonal axes. Likewise, it will have at least one rotational degree-of-freedom about each of three orthogonal axes. The effective modal mass calculation for a general system is shown by the example in Appendix A. The example is from a real-world problem.

Aside

An alternate definition of the participation factor is given in Appendix B.

References

1. M. Papadrakakis, N. Lagaros, V. Plevris; Optimum Design of Structures under Seismic Loading, European Congress on Computational Methods in Applied Sciences and Engineering, Barcelona, 2000.

2. T. Irvine, The Generalized Coordinate Method For Discrete Systems, Vibrationdata, 2000.

3. W. Thomson, Theory of Vibration with Applications 2nd Edition, Prentice Hall, New Jersey, 1981.

4. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Rev M, Vibrationdata, 2010.

5. T. Irvine, Rod Response to Longitudinal Base Excitation, Steady-State and Transient, Rev B, Vibrationdata, 2009.

APPENDIX A

Equation of Motion

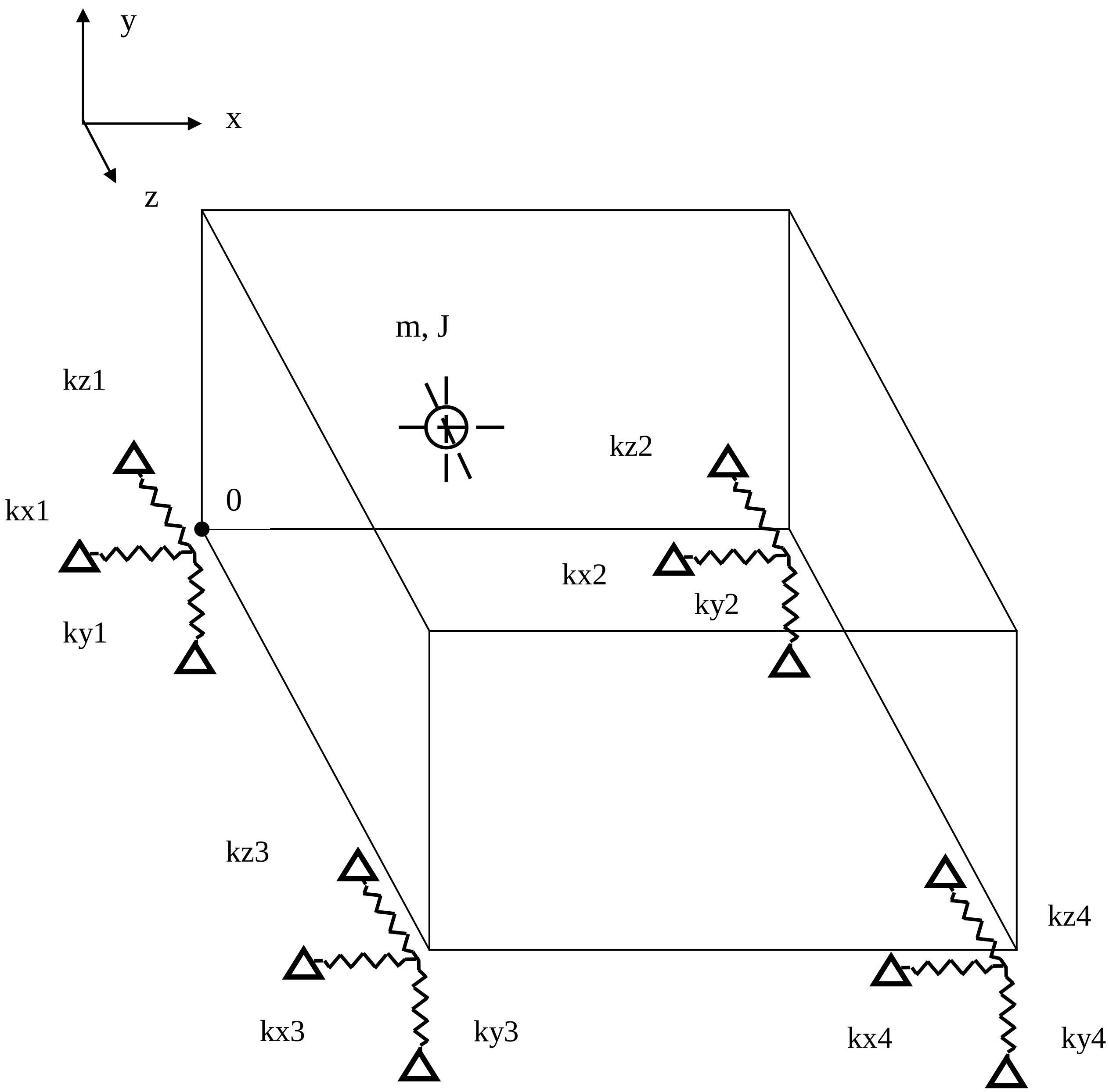

Figure A-1. Isolated Avionics Component Model

The mass and inertia are represented at a point with the circle symbol. Each isolator is modeled by three orthogonal DOF springs. The springs are mounted at each corner. The springs are shown with an offset from the corners for clarity. The triangles indicate fixed constraints. "0" indicates the origin.

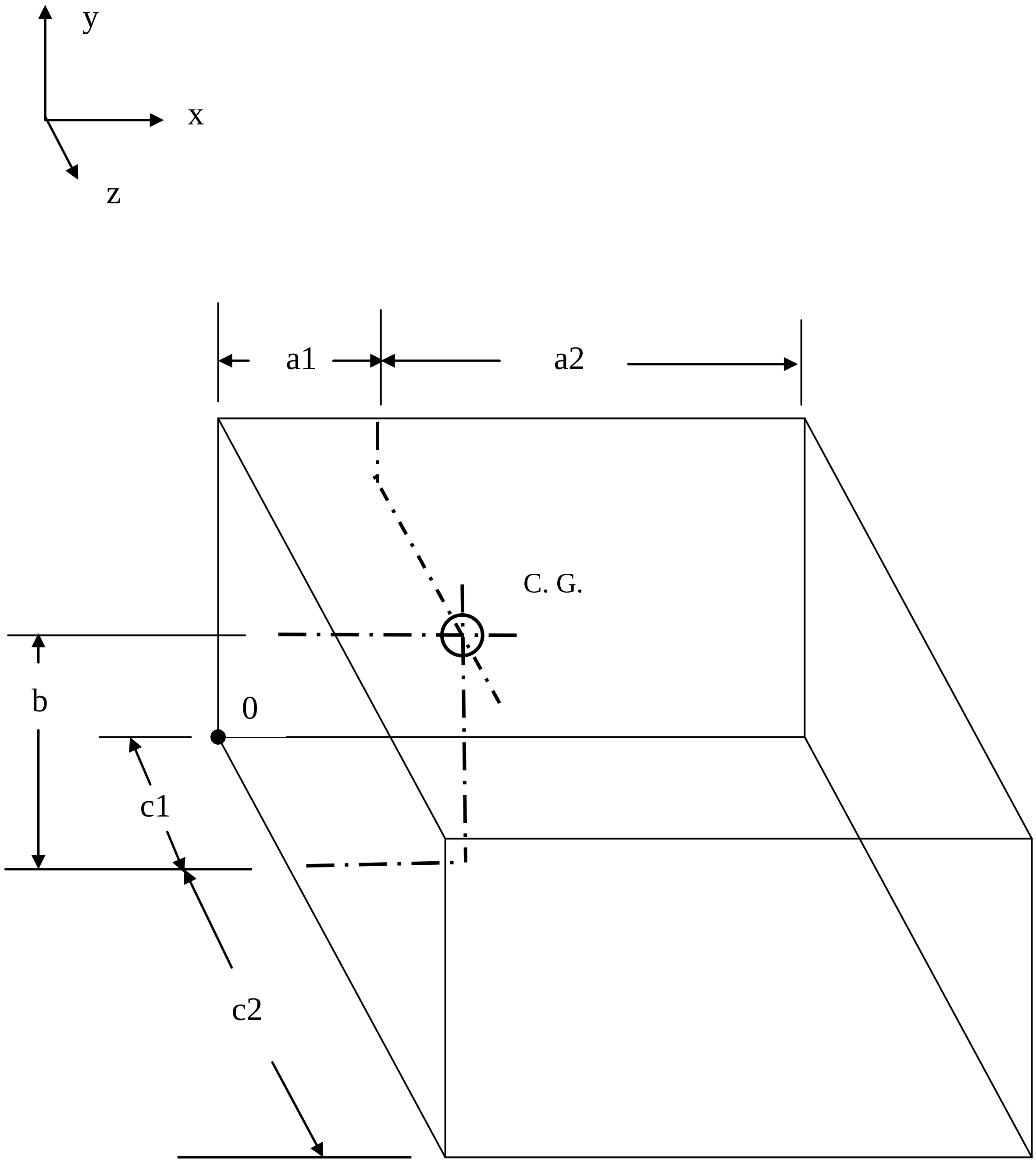

Figure A-2. Isolated Avionics Component Model with Dimensions

All dimensions are positive as long as the C.G. is "inside the box." At least one dimension will be negative otherwise.

The mass and stiffness matrices are shown in upper triangular form due to symmetry.

$$\underline{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ & m & 0 & 0 & 0 & 0 \\ & & m & 0 & 0 & 0 \\ & & & J_x & 0 & 0 \\ & & & & J_y & 0 \\ & & & & & J_z \end{bmatrix}$$

(A-1)

$$\underline{K} = \begin{bmatrix} 4k_x & 0 & 0 & 0 & 2k_x(-c_1 + c_2) & 4k_x b \\ & 4k_y & 0 & 2k_y(c_1 - c_2) & 0 & 2k_y(-a_1 + a_2) \\ & & 4k_z & -4k_z b & 2k_z(a_1 - a_2) & 0 \\ & & & 4k_z b^2 + 2k_y\left(c_1^2 + c_2^2\right) & 2k_z(-a_1 + a_2)b & k_y(-a_1 + a_2)(c_1 - c_2) \\ & & & & 4k_x c_2^2 + 2k_z\left(a_1^2 + a_2^2\right) & 2k_x(-c_1 + c_2)b \\ & & & & & 4k_x b^2 + 2k_y\left(a_1^2 + a_2^2\right) \end{bmatrix}$$

(A-2)

The equation of motion is

$$\underline{M}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\\ \ddot{\alpha}\\ \ddot{\beta}\\ \ddot{\theta}\end{bmatrix}+\underline{K}\begin{bmatrix}x\\ y\\ z\\ \alpha\\ \beta\\ \theta\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \tag{A-3}$$

The variables $\alpha, \beta,$ and θ represent rotations about the X, Y, and Z axes, respectively.

Example

A mass is mounted to a surface with four isolators. The system has the following properties.

M	=	4.28 lbm
Jx	=	44.9 lbm in^2
Jy	=	39.9 lbm in^2
Jz	=	18.8 lbm in^2
kx	=	80 lbf/in
ky	=	80 lbf/in
kz	=	80 lbf/in
a1	=	6.18 in
a2	=	-2.68 in
b	=	3.85 in
c1	=	3. in
c2	=	3. in

Let $\bar{r}$ be the influence matrix which represents the displacements of the masses resulting from static application of unit ground displacements and rotations. *The influence matrix for this example is the identity matrix provided that the C.G is the reference point.*

$$\bar{r} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A-4}$$

The coefficient matrix $\bar{L}$ is

$$\bar{L} = \phi^{T} M \, \bar{r} \tag{A-5}$$

The modal participation factor matrix Γ_{i} for mode i at dof j is

$$\Gamma_{ij} = \frac{\bar{L}_{ij}}{\hat{m}_{ii}} \tag{A-6}$$

Each $\hat{m}_{ii}$ coefficient is 1 if the eigenvectors have been normalized with respect to the mass matrix.

The effective modal mass $m_{eff,i}$ vector for mode i and dof j is

$$m_{eff,ij} = \frac{\left[\bar{L}_{ij}\right]^2}{\hat{m}_{ii}} \tag{A-7}$$

The natural frequency results for the sample problem are calculated using the program: six_dof_iso.m.

The results are given in the next pages.

```
six_dof_iso.m   ver 1.2  March 31, 2005

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 This program finds the eigenvalues and eigenvectors for a
 six-degree-of-freedom system.

 Refer to six_dof_isolated.pdf for a diagram.

 The equation of motion is:   M (d^2x/dt^2) + K x = 0

 Enter m (lbm)
 4.28
 Enter Jx (lbm in^2)
 44.9
 Enter Jy (lbm in^2)
 39.9
 Enter Jz (lbm in^2)
 18.8

 Note that the stiffness values are for individual springs

 Enter kx (lbf/in)
 80

 Enter ky (lbf/in)
 80

 Enter kz (lbf/in)
 80

 Enter a1 (in)
 6.18

 Enter a2 (in)
 -2.68

 Enter b (in)
 3.85

 Enter c1 (in)
 3
```

```
Enter c2 (in)
3


 The mass matrix is

m =

    0.0111         0         0         0         0         0
         0    0.0111         0         0         0         0
         0         0    0.0111         0         0         0
         0         0         0    0.1163         0         0
         0         0         0         0    0.1034         0
         0         0         0         0         0    0.0487


 The stiffness matrix is

k =

  1.0e+004 *

    0.0320         0         0         0         0    0.1232
         0    0.0320         0         0         0   -0.1418
         0         0    0.0320   -0.1232    0.1418         0
         0         0   -0.1232    0.7623   -0.5458         0
         0         0    0.1418   -0.5458    1.0140         0
    0.1232   -0.1418         0         0         0    1.2003


 Eigenvalues

lambda =

  1.0e+005 *

    0.0213    0.0570    0.2886    0.2980    1.5699    2.7318
```

Natural Frequencies =

1.	7.338 Hz
2.	12.02 Hz
3.	27.04 Hz
4.	27.47 Hz
5.	63.06 Hz
6.	83.19 Hz

Modes Shapes (rows represent modes)

	x	y	z	alpha	beta	theta
1.	5.91	-6.81	0	0	0	-1.42
2.	0	0	8.69	0.954	-0.744	0
3.	7.17	6.23	0	0	0	0
4.	0	0	1.04	-2.26	-1.95	0
5.	0	0	-3.69	1.61	-2.3	0
6.	1.96	-2.25	0	0	0	4.3

Participation Factors (rows represent modes)

	x	y	z	alpha	beta	theta
1.	0.0656	-0.0755	0	0	0	-0.0693
2.	0	0	0.0963	0.111	-0.0769	0
3.	0.0795	0.0691	0	0	0	0
4.	0	0	0.0115	-0.263	-0.202	0
5.	0	0	-0.0409	0.187	-0.238	0
6.	0.0217	-0.025	0	0	0	0.21

Effective Modal Mass (rows represent modes)

	x	y	z	alpha	beta	theta
1.	0.0043	0.00569	0	0	0	0.0048
2.	0	0	0.00928	0.0123	0.00592	0
3.	0.00632	0.00477	0	0	0	0
4.	0	0	0.000133	0.069	0.0408	0
5.	0	0	0.00168	0.035	0.0566	0
6.	0.000471	0.000623	0	0	0	0.0439

Total Modal Mass

0.0111	0.0111	0.0111	0.116	0.103	0.0487

APPENDIX B

The following definition is taken from Reference 3. Note that the mode shape functions are unscaled. Hence, the participation factor is unscaled.

Modal Participation Factor

Consider a beam of length L loaded by a distributed force p(x,t).

Consider that the loading per unit length is separable in the form

$$p(x,t) = \frac{P_o}{L} p(x) f(t) \tag{B-1}$$

The modal participation factor Γ_i for mode i is defined as

$$\Gamma_i = \frac{1}{L} \int_0^L p(x) \phi_i(x) \, dx \tag{B-2}$$

where

$\phi_i(x)$ is the normal mode shape for mode i

APPENDIX C

The following convention appears to be more useful than that given in Appendix B.

Modal Participation Factor for a Beam

Let

$Y_n(x)$ = mass-normalized eigenvectors

$m(x)$ = mass per length

The participation factor is

$$\Gamma_n = \int_0^L m(x) Y_n(x) dx \tag{C-1}$$

The effective modal mass is

$$m_{eff,n} = \frac{\left[\int_0^L m(x) Y_n(x) dx \right]^2}{\int_0^L m(x) [Y_n(x)]^2 dx} \tag{C-2}$$

The eigenvectors should be normalized such that

$$\int_0^L m(x) [Y_n(x)]^2 dx = 1 \tag{C-3}$$

Thus,

$$m_{eff,n} = [\Gamma_n]^2 = \left[\int_0^L m(x) Y_n(x) dx \right]^2 \tag{C-4}$$

APPENDIX D

Effective Modal Mass Values for Bernoulli-Euler Beams

The results are calculated using formulas from Reference 4. The variables are

E	=	is the modulus of elasticity
I	=	is the area moment of inertia
L	=	is the length
ρ	=	is (mass/length)

Table D-1. Bending Vibration, Beam Simply-Supported at Both Ends

Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	$4\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
3	$9\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
4	$16\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
5	$25\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$
6	$36\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	0	0
7	$49\frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}$	$\frac{2}{7\pi}\sqrt{2\rho L}$	$\frac{8}{49\pi^2}\rho L$

95% of the total mass is accounted for using the first seven modes.

Table D-2. Bending Vibration, Fixed-Free Beam			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$\left[\frac{1.87510}{L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.7830\sqrt{\rho L}$	$0.6131\ \rho L$
2	$\left[\frac{4.69409}{L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.4339\sqrt{\rho L}$	$0.1883\ \rho L$
3	$\left[\frac{5\pi}{2L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.2544\sqrt{\rho L}$	$0.06474\ \rho L$
4	$\left[\frac{7\pi}{2L}\right]^2 \sqrt{\frac{EI}{\rho}}$	$0.1818\sqrt{\rho L}$	$0.03306\ \rho L$

90% of the total mass is accounted for using the first four modes.

APPENDIX E

Rod, Longitudinal Vibration

The results are taken from Reference 5.

Table E-1. Longitudinal Vibration of a Rod, Fixed-Free			
Mode	Natural Frequency ω_n	Participation Factor	Effective Modal Mass
1	$0.5\pi\ c/L$	$\frac{2}{\pi}\sqrt{2\rho L}$	$\frac{8}{\pi^2}\rho L$
2	$1.5\pi\ c/L$	$\frac{2}{3\pi}\sqrt{2\rho L}$	$\frac{8}{9\pi^2}\rho L$
3	$2.5\pi\ c/L$	$\frac{2}{5\pi}\sqrt{2\rho L}$	$\frac{8}{25\pi^2}\rho L$

The longitudinal wave speed c is

$$c = \sqrt{\frac{E}{\rho}} \tag{E-1}$$

93% of the total mass is accounted for by using the first three modes.